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GUGGENHEIM AERONAUTICAL LABORATORY

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HYPERSONIC RESEARCH PROJECT

Memorandum No. 58
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KINETIC THEORY DESCRIPTION OF PLANE, COMPRESSIBLE COUETTE FLOW

by
Lester Lees and Chung-Yen Liu

ARMY ORDNANCE CONTRACT NO. DA-04-495-Ord-1960

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ABSTRACT

By utilizing the two-stream Maxwellian in Maxwell's integral equations of transfer we are able to find a closed-form solution of the problem of compressible plane Couette flow over the whole range of gas density from free molecule flow to atmospheric. The ratio of shear stress to the product of ordinary viscosity and velocity gradient, which is unity for a Newtonian fluid, here depends also on the gas density, the plate temperatures and the plate spacing. For example, this ratio decreases rapidly with increasing plate Mach number when the plate temperatures are fixed. On the other hand, at a fixed Mach number based on the temperature of one plate, this ratio approaches unity as the temperature of the other plate increases. Similar remarks can be made for the ratio of heat flux to the product of ordinary heat conduction coefficient and temperature gradient.

The effect of gas density on the skin friction and heat transfer coefficients is described in terms of a single rarefaction parameter, which amounts to evaluating gas properties at a certain "kinetic temperature" defined in terms of plate Mach number and plate temperature ratio. One interesting result is the effect of plate temperature on velocity "slip". In the Navier-Stokes regime most of the gas follows the hot plate, because the gas viscosity is larger there. As the gas density decreases the situation is reversed, because the velocity slip is larger at the hot plate than at the cold plate. In the limiting case of a highly rarefied gas most of the gas follows the cold plate.

Limitations of the present six-moment approximation at high plate Mach numbers are discussed and it is concluded that an eight-moment approximation would eliminate these difficulties. The results obtained in this simple geometry suggest certain conclusions about hypersonic flow over solid bodies when the surface temperature is much lower than the kinetic temperature.

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LIST OF SYMBOLS

a_1, a_2	functions in two-stream Maxwellian for eight-moment approximation
A_2	= 1.3682, value of scattering integral
b	impact parameter, or perpendicular distance from particle "i" to initial trajectory of particle "j"
b_i	integration constant, i = 1, 2, 3, 4. (See Appendix.)
$\vec{c}$	relative particle velocity, $\vec{\xi} - \vec{u}$
$\tilde{c}$	mean molecular speed, $\sqrt{8 \mathcal{R} T/\pi}$
c_i	component of relative particle velocity in i^{th} direction
c_p, c_v	heat capacity at constant pressure and constant volume, respectively
C_D	skin-friction coefficient, $p_{xy}/(\frac{1}{2}\rho_{II} U^2)$
C_H	Stanton number, $q_y / \left[\rho_{II} c_p U (T_{II} - T_I)\right]$
d	distance between lower and upper plates
f, f_1	velocity distribution functions for "probe" and colliding particles, respectively
f_1, f_2	components of two-stream Maxwellian
f_{max}	local full-range Maxwellian
F	interparticle force, also function defined by Eq. (33)
G	function defined by the relation $\bar{n}_1\sqrt{\bar{T}_1} = \bar{n}_2\sqrt{\bar{T}_2} = [G]^{-1}$
k	Boltzmann constant
k_c	"classical" thermal conductivity
K	constant in expression for inverse fifth-power force law, $F = (m_1 m_2 K)/r$
L	square root of plate temperature ratio, $\sqrt{T_I/T_{II}}$
m	mass of a particle
M	Mach number, $U/\sqrt{\gamma \mathcal{R} T_{II}}$
$\tilde{M}$	"proper" Mach number, $\tilde{M}^2 = U^2/\gamma \mathcal{R} \sqrt{T_I T_{II}}$

n	particle number density, per unit volume
n_1, n_2	number density functions in two-stream Maxwellian
p	$nkT = \rho \mathcal{R} T$
p_{ii}	defined by the relation $P_{ii} = -p + p_{ii}$
$p_{ij} \equiv P_{ij}$	shear stress, $i \neq j$, $p_{ij} = -m \int fc_i c_j d\vec{\xi}$
P_{ii}	normal stress, $P_{ii} = -m \int fc_i^{\,2} d\vec{\xi}$
Pr	Prandtl number, $c_p \mu_c / k_c$
q_y	heat flux in y-direction
Q	arbitrary function of particle velocity
ΔQ	change in Q produced by collisions
r	distance between two particles
$\vec{R}$	radius vector
Re	Reynolds number, $\rho_{II} Ud / \mu_{II}$
$\mathcal{R}$	gas constant, k/m
s_n	non-dimensional relative velocity in the normal direction
t	time
T	absolute temperature, $3/2\, n\, k\, T = m \int (c^2/2)\, f\, d\vec{\xi}$
T_1, T_2	temperature functions in two-stream Maxwellian
$\vec{u}$	mean velocity vector, $\rho\vec{u} = m \int f\vec{\xi}\, d\vec{\xi}$
u, v	components of mean velocity parallel to x- and y- axes, respectively
$\vec{u}_\infty$	free stream velocity vector
$\vec{u}_1, \vec{u}_2$	vector velocity functions in two-stream Maxwellian
u_i	component of mean velocity in i^{th} direction
U	relative plate velocity
V	relative velocity between two interacting particles = $\lvert \xi_1 - \xi \rvert$
x, y	coordinates along and normal to plates
x_i	coordinate in i^{th} direction

a_i	integration constant, $i = 1, 2, 3, 4, 5$
β	quantity defined by the relation $\beta = (8/15)\sqrt{2/\pi\gamma}\ (Re/M)$
γ	ratio of specific heats, c_p/c_v
ε	angle between plane of the orbit and plane containing the original relative velocity and the x-axis in a binary collision
λ	Maxwell mean free path
λ_i	$i = 1, 2, 3$, functions defined by Eqs. (35a) and (37)
Λ	Pohlhausen parameter
μ_c	"classical" viscosity coefficient
$\tilde{\mu}$	viscosity coefficient = $p_{xy}/(du/dy)$
$\vec{\xi}$	vector particle velocity, $\xi^2 = \lvert\xi\rvert^2$
$d\vec{\xi}$	$d\xi_i\ d\xi_j\ d\xi_k$
ξ_j	component of particle velocity in j^{th} direction
ξ_1	velocity of colliding particles
ρ	nm, mass density, $\rho = \int m f\, d\vec{\xi}$

The subscripts "1" and "2" generally denote the two components of the two-stream Maxwellian, and the subscripts "I" and "II" refer to quantities given at the upper and lower plates respectively. A prime denotes quantities evaluated after a collision, while unprimed quantities refer to conditions before a collision. The subscript "o" denotes free molecular flow conditions, the subscript "w" denotes surface values, the subscript "∞" denotes free stream quantities far ahead of a body, and the subscript "n" denotes quantities normal to the surface.

I. INTRODUCTION

In principle, the Maxwell-Boltzmann integro-differential equation for the single particle velocity distribution function is fully capable of describing the flow of a monatomic gas over the whole range of gas densities from "free-molecule" flow to the classical Navier-Stokes regime [(2), (8), (10)]. However the formidable difficulties involved in constructing solutions of this equation are too well known to require repetition here (6). Fortunately, in fluid mechanics one is not particularly interested in the velocity distribution function itself, but in certain lower moments of this function, such as mean velocity, shear stress, etc. Recognizing this fact, Maxwell (11) converted the original Maxwell-Boltzmann equation into an integral equation of transfer, or moment equation, for any quantity Q that is a function only of the components of the particle velocity. In the absence of external forces Maxwell's integral equation takes the following form in a rectangular Cartesian coordinate system:*

$$(\partial/\partial t)\left(\int f\, Q\, d\vec{\xi}\right) + \sum_i (\partial/\partial x_i)\left(\int f\, \xi_i\, Q\, d\vec{\xi}\right) = \Delta Q \quad , \tag{1}$$

where ΔQ represents the time rate of change of Q produced by particle collisions, and is given by

$$\Delta Q = \iiiint (Q' - Q)\, f\, f_1\, V\, d\vec{\xi}\, d\vec{\xi}_1\, b\, d\, b\, d\, \varepsilon \quad . \tag{2}$$

Actually Maxwell employed a special form of the distribution function, but an important advantage of Eq. (1) is just the fact that it permits a large amount of flexibility in the choice of f. The distribution

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* Maxwell's integral equation including external forces and coordinate system curvature is given in Reference 10.

function can be expressed in terms of a certain number of arbitrary functions of space and time, selected in such a way that essential physical features of the problem are introduced. Of course the proper number of moments (Q's) must be taken to insure that a complete set of first-order partial differential equations is obtained for these undetermined functions. As shown by Maxwell (11) the ordinary gas dynamic conservation equations are obtained regardless of the choice of f by taking Q to be successively the collisional invariants of mass, momentum, and energy $\left[Q = m,\ m\xi_i,\ m\xi^2/2\right]$, for which $\Delta Q = 0$. The number of additional moments (and arbitrary functions) employed depends on the degree of detail desired, and also on the relative magnitude of these additional moments (Section III. E).

Clearly this procedure amounts to satisfying the Maxwell-Boltzmann equation in a certain average sense, rather than point-by-point, just as one does in the more familiar Kármán-Pohlhausen method for boundary layer flows (14) and its extension by Tani (13). The distribution function employed should be regarded as a suitable weighting function which is not in general an "exact" solution of the original Maxwell-Boltzmann equation. Thus, there is no need to retain the undesirable rigidity inherent in a polynomial of Chapman-Enskog type, as in Grad's method (5). In fact, Mott-Smith (12) found that a distribution function consisting of the sum of two full-range Maxwellians is quite suitable for a rough description of the structure of a strong, steady, normal shock wave. A careful study of shear flows in rarefied gases and of the difficulties encountered with Grad's thirteen moment approximation (1) shows that the following basic requirements must be satisfied by the distribution function employed in Maxwell's moment method (10):

(1) It must have the "two-sided" character that is an essential feature of highly rarefied gas flows, and especially of non-linear rarefied flows;

(2) It must be capable of providing a smooth transition from rarefied flows to the classical Navier-Stokes regime;

(3) It should lead to the simplest possible set of differential equations and boundary conditions consistent with (1) and (2).

Of course the class of distribution functions satisfying requirements (1) and (2) is very large. In Reference 10, one of the present authors introduced the "two-stream" Maxwellian, which is probably one of the simplest such functions, as a natural generalization of the situation for free-molecule flow. In body coordinates all outwardly directed particle velocity vectors lying within the "cone of influence" (Region 1 in Figure 1) are described by the function $f = f_1$, where

$$f_1 = \frac{n_1(\vec{R}, t)}{[2\pi \mathcal{R} T_1(\vec{R}, t)]^{3/2}} \exp\left\{ - \frac{[\vec{\xi} - \vec{u}_1(\vec{R}, t)]^2}{2 \mathcal{R} T_1(\vec{R}, t)} \right\} \tag{3a}$$

In Region 2 (all other $\vec{\xi}$)

$$f = f_2 = \frac{n_2(\vec{R}, t)}{[2\pi \mathcal{R} T_2(\vec{R}, t)]^{3/2}} \exp\left\{ - \frac{[\vec{\xi} - \vec{u}_2(\vec{R}, t)]^2}{2 \mathcal{R} T_2(\vec{R}, t)} \right\} \tag{3b}$$

where n_1, n_2, T_1, T_2, $\vec{u}_1$, $\vec{u}_2$ are ten initially undetermined functions of $\vec{R}$ and t. In the limiting case of free-molecule flow the distribution function described by Eqs. (3a) and (3b) is an exact solution for completely diffuse reemission, provided that $\vec{u}_2 = \vec{u}_\infty$, $n_2 = n_\infty$, $T_2 = T_\infty$, $\vec{u}_1 = \vec{u}_w$, $T_1 = T_w$, and n_1 is the function determined by the boundary condition on the normal velocity at the body surface. In the present method the variation of these ten functions with $\vec{R}$ and t is a measure of the effect of particle collisions in the gas, as determined by Maxwell's moment equations [Eqs. (1) and (2)]. Thus, one gives up once and for all the search for "higher order" macroscopic equations in terms of the mean quantities, such as the Burnett equations, Grad's equations, etc. Once these ten functions are determined, the mean quantities are obtained by utilizing the distribution function defined by Eqs. (3a) and (3b), as in any true statistical approach.

One important advantage of the two stream Maxwellian is that the surface boundary conditions are easily incorporated into the analysis [Requirement (3) above]. For example, for completely diffuse reemission, the reemitted particles have a Maxwellian velocity distribution corresponding to T_w, by definition, and the mean velocity of the reemitted particles is identical with the local surface velocity. Thus [Eq. (3a)], $\vec{u}_1(\vec{R}, t) = \vec{u}_w$ and $T_1(\vec{R}, t) = T_w$ when $\vec{R} = \vec{R}_w$. When there is no net mass transfer at the surface an additional boundary condition must be satisfied which is similar to the usual free-molecule flow condition, except that now $\vec{u}_2 \neq \vec{u}_\infty$ in general:

$$n_1 \sqrt{\mathcal{R} T_1} = n_1 \sqrt{\mathcal{R} T_w} = n_2 \sqrt{\mathcal{R} T_2}\; C(-s_{2_n}) , \qquad (4)$$

where

$$C(s_n) = e^{-s_n^2} + \sqrt{\pi}\; s_n (1 + \operatorname{erf} s_n) . \qquad (5)$$

Here

$$s_{2_n} = \frac{(u_{2_n} - u_{w_n})}{\sqrt{2 \mathcal{R} T_2}} ,$$

where u_{2_n} and u_{w_n} are the normal components of $\vec{u}_2$ and $\vec{u}_w$, respectively. In considering the uniform rectilinear motion of a finite body in a fluid of infinite extent the following boundary conditions must also be imposed (in body coordinates):

$$\vec{u}_2 \longrightarrow \vec{u}_\infty \ , \ T_2 \longrightarrow T_\infty \ , \ n_2 \longrightarrow n_\infty \ , \ \text{as } x \longrightarrow -\infty \ .$$

As an illustration the present method was applied in Reference 10 to linearized plane Couette flow and to the linearized form of Rayleigh's problem. But plane, parallel flows at low Mach number with small temperature differences cannot provide a serious test of any method that is supposed to be general. In this paper we apply the present technique to steady, plane compressible Couette flow, in order to study the effects of large temperature differences and dissipation in the simplest possible geometry. In Section II. A. the basic equations and boundary conditions for this problem are formulated. In order to simplify the work the particles are supposed to obey Maxwell's inverse fifth-power law of repulsion, but this restriction is not an essential one. Solutions are obtained first for arbitrary temperature ratio between the two plates, but $M^2 \longrightarrow << 1$, (Section II. B.), and then similar methods are employed for the case of arbitrary Mach number and temperature ratio (Sections II. C. and II.D..). In Section III we utilize the calculated behavior of the velocity and temperature profiles and other mean quantities in this problem to gain some insight into the effect of Mach number and the ratio of plate temperatures on the nature of the transition from free-molecule flow to the classical Navier-Stokes regime.

II. MAXWELL'S MOMENT METHOD FOR PLANE COMPRESSIBLE COUETTE FLOW

II. A. Formulation of the Problem

II. A. 1. Differential Equations

Maxwell's moment method is applied to the problem of the steady flow generated by the relative shearing motion of two infinite parallel flat plates. The upper plate moves with velocity + $U/2$ in its own plane at $y = d/2$ and is held at temperature T_I , while the lower plate at $y = - d/2$ moves parallel to the upper plate with velocity - $U/2$, and is kept at temperature T_{II} [Figure 2] . The only independent variable in this problem is the coordinate normal to the plates, y ; thus, Eq. (1) reduces to

$$d/dy \left(\int f \xi_y Q \, d\vec{\xi} \right) = \Delta Q \quad . \tag{6}$$

By taking Q to be the collisional invariants m, $m \xi_x$, $m \xi_y$, and $m \xi^2/2$, successively, four equations are obtained from Eq. (6), corresponding to the ordinary gas dynamic conservation equations. For these moments $\Delta Q = 0$, and

$$\int f \xi_y Q \, d\vec{\xi} = \text{constant} \quad . \tag{7}$$

According to kinetic theory,

$$\rho u_i = \int m f \xi_i \, d\vec{\xi} \quad , \tag{8a}$$

where

$$\rho = \int m f \, d\vec{\xi} \quad . \tag{8b}$$

Thus the first of Eqs. (7) with $Q = m$ is just the ordinary equation of continuity for this problem, namely,

$$\rho\, v = \text{constant} \quad . \tag{9a}$$

But

$$v(-d/2) = v(d/2) = 0 \text{ , so that } v(y) \equiv 0 \quad . \tag{9b}$$

By definition $P_{ij} = -m \int f\, c_i\, c_j\, d\vec{\xi}$,

where $\vec{c}$ is the intrinsic or relative velocity $\vec{\xi} - \vec{u}$. Here

$$P_{ij} = p_{ij} \ , \quad i \neq j \quad ,$$

$$P_{ii} = -p + p_{ii} \ , \quad i = j \quad ,$$

where $p = -\sum (P_{ii}/3) = \rho\, \mathcal{R}\, T = (2/3) \int m\, f\, (c^2/2)\, d\vec{\xi}$.

With $Q = m\, \xi_x$ and $m\, \xi_y$, one obtains [Eq. (7) and (9b)]

$$p_{xy} = \text{constant} \tag{9c}$$

and

$$P_{yy} = \text{constant, respectively.} \tag{9d}$$

Similarly, by taking $Q = m\, \xi^2/2$ in Eq. (7) , and recognizing that $\xi_x = c_x + u$, $\xi_y = c_y$, $\xi_z = c_z$ and $v \equiv 0$ in this problem, one finds as expected that

$$q_y - p_{xy}\, u = \text{constant} \tag{9e}$$

where

$$q_y = m \int f\, c_y\, c^2/2\, d\vec{\xi} \text{ , by definition.}$$

In this case the "two-stream Maxwellian" [Eqs. (3a) and (3b)] takes the following form (Figure 2) :

For $\xi_y < 0$,

$$f = f_1 = n_1(y) \frac{1}{[2\pi \mathcal{R} T_1(y)]^{3/2}} \exp\left\{ - \frac{[\xi_x - u_1(y)]^2 + \xi_y^{\,2} + \xi_z^{\,2}}{2 \mathcal{R} T_1(y)} \right\} \ . \tag{10a}$$

For $\xi_y > 0$,

$$f = f_2 \quad , \tag{10b}$$

where f_2 is a similar generalized Maxwellian containing the functions $n_2(y)$, $T_2(y)$, $u_2(y)$. Two independent moment equations in addition to the four represented by Eqs. (7) [or Eqs. (9a) - (9e)] are required to determine these six arbitrary functions of y. Of course these additional moments can be chosen quite arbitrarily. Because of our special interest in the shear stress and normal heat flux in this problem we take $Q_5 = m\, \xi_x \xi_y$ and $Q_6 = m\, \xi_y\, (\xi^2/2)$. [See, however, Section III. E.]

Once the two-stream Maxwellian is selected for f, the collision integral ΔQ [Eq. (2)] can be evaluated for any arbitrary law of force between the particles. For simplicity we utilize Maxwell's inverse fifth-power force law $F = \frac{m_1 m_2 K}{r^5}$. With this choice the relative velocity $V = | \vec{\xi}_1 - \vec{\xi} |$ is eliminated from the collision integral, and ΔQ for the lower moments contains the components of the heat flux vector and the shear stress tensor [(7), (10), (11)]. To be specific, for $Q_5 = m\, \xi_j \xi_k$, one finds

$$\Delta Q_5 = (3/2)\ A_2 \sqrt{2\, m\, K}\ \ n\ p_{jk}\ ; \tag{11}$$

while if $Q_6 = m\, \xi_j\, (\xi^2/2)$, one has

$$\Delta Q_6 = (3/2)\ A_2 \sqrt{2\, m\, K}\ \ n \left[-(2/3)\, q_j + \sum_k p_{jk}\, u_k \right] . \tag{12}$$

[Here $A_2 = 1.3682$ is the value of the scattering integral found by Maxwell (11).] Both of these results are independent of the choice of f.

Now the ordinary or "classical" coefficient of viscosity for Maxwell particles based on the local full-range Maxwellian is given by the expression

$$\mu_c = \frac{k\, T}{(3/2)\ A_2 \sqrt{2\, m\, K}} , \tag{13}$$

where k is the Boltzmann constant. Therefore,

$$(3/2)\ A_2 \sqrt{2\,m\,K}\ \ n = (p/\mu_c)\ . \quad * \tag{14}$$

Thus the two moment equations supplementing Eqs. (7) are as follows [Eqs. (6), (11), and (12)] :

$$(d/dy)\left(\int m\,f\,\xi_x\,\xi_y^{\,2}\,d\vec{\xi}\,\right) = (p/\mu_c)\,p_{xy} \tag{15}$$

$$(d/dy)\left(\int m\,f\,\xi_y^{\,2}\,\xi^2/2\ \ d\vec{\xi}\right) = (p/\mu_c)\left[-(2/3)\,q_y + p_{xy}\,u + \cancel{p_{yy}\,v}^{\,0}\right]. \tag{16}$$

If the local full-range Maxwellian

$$f_{max}(\vec{\xi},\ y) = \frac{n}{(2\pi \mathcal{R} T)^{3/2}}\ \exp\left(-\ \frac{c^2}{2\,\mathcal{R}T}\right)$$

is introduced into the left-hand sides of Eqs. (15) and (16) one obtains the familiar relations

$$\mu_c\,(du/dy) = p_{xy}$$

$$-\,(3/2)\ c_p\,\mu_c\,(dT/dy) = -\,k_c\,(dT/dy) = q_y\quad .$$

In fact this approximation corresponds exactly to the first step of the Chapman-Enskog expansion procedure(see for example Reference 10). But in general $f \neq f_{max}$, so that $p_{xy} \neq \mu_c$ (du/dy) and $q_c \neq -\,k_c$ (dT/dy).

Six equations for the six arbitrary functions of y appearing in the two stream Maxwellian are obtained by substituting this f [Eqs. (3a) and (3b)] into Eqs. (7) and Eqs. (15) and (16). All moments and mean flow quantities are evaluated as follows:

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* The ordinary coefficient of viscosity μ_c is introduced here mainly for convenience. It must be emphasized that $\mu_c \neq p_{jk} \,/\, \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)$, except in the limiting case $Re/M \longrightarrow \infty$, which corresponds to the classical Navier-Stokes regime (Sections II. B and II. C).

$$< \phi > = \int \phi f \, d\vec{\xi} = \int_{-\infty}^{+\infty} \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \phi f_1 \, d\xi_x \, d\xi_y \, d\xi_z + \int_{-\infty}^{+\infty} \int_{0}^{\infty} \int_{-\infty}^{+\infty} \phi f_2 \, d\xi_x \, d\xi_y \, d\xi_z \quad .$$

For example,

$$\rho(y) = < m > = (m/2) \left[n_1(y) + n_2(y) \right] = m \, n(y) \tag{17}$$

$$u(y) = (1/\rho) < m \xi_x > = \frac{n_1(y) \, u_1(y) + n_2(y) \, n_2(y)}{n_1(y) + n_2(y)} \quad . \tag{18}$$

Since it is more convenient to work with non-dimensional quantities, we select n_{II}, U, T_{II}, d as the characteristic number density, velocity, temperature and length, respectively. A Mach number and Reynolds number are introduced based on these characteristic quantities

$$M = U / \sqrt{\gamma \mathcal{R} T_{II}}$$

$$Re = \frac{m \, n_{II} \, U \, d}{(\mu_c)_{II}} \quad ,$$

where $(\mu_c)_{II}$ denotes the ordinary viscosity coefficient evaluated at temperature T_{II}. The parameter Re/M is inversely proportional to the ratio of mean free path, λ_{II}, to the characteristic length d, and this parameter characterizes the density level of the gas. In fact

$$Re/M = \sqrt{\pi \gamma / 2} \; (d/ \lambda_{II}) \quad .$$

Let normalized quantities be denoted by a bar superscript. Then the non-dimensional governing equations are as follows:

<u>Continuity</u>

$$\bar{n}_1 \sqrt{\bar{T}_1} = \bar{n}_2 \sqrt{\bar{T}_2} \tag{19a}$$

Momentum

$$\bar{n}_1 \sqrt{\bar{T}_1} \quad (\bar{u}_2 - \bar{u}_1) = a_1 \tag{19b}$$

$$\bar{n}_1 \bar{T}_1 + \bar{n}_2 \bar{T}_2 = a_2 \tag{19c}$$

Energy

$$\bar{n}_1 \sqrt{\bar{T}_1} \left[\bar{T}_2 - \bar{T}_1 + (\gamma M^2/4)(\bar{u}_2^{\,2} - \bar{u}_1^{\,2})\right] = a_2 a_3 \tag{19d}$$

Stress

$$(d/d\bar{y})(\bar{n}_1\bar{u}_1\bar{T}_1 + \bar{n}_2\bar{u}_2\bar{T}_2) + \frac{1}{\sqrt{2\pi\gamma}} (\mathrm{Re}/M)\, a_1 (\bar{n}_1 + \bar{n}_2) = 0 \tag{19e}$$

Heat Flux

$$\begin{aligned}
&(d/d\bar{y})(\bar{n}_1\bar{T}_1^{\,2} + \bar{n}_2\bar{T}_2^{\,2}) + (\gamma M^2/5)(d/d\bar{y})\left[\bar{n}_1\bar{T}_1\bar{u}_1^{\,2} + \bar{n}_2\bar{T}_2\bar{u}_2^{\,2}\right] \\
&- (2/5)\,\gamma M^2\, \bar{u}\, (d/d\bar{y})\, (\bar{n}_1\bar{T}_1\bar{u}_1 \qquad + \bar{n}_2\bar{T}_2\bar{u}_2) \\
&- (4/15)\sqrt{\gamma/2\pi}\ (\mathrm{Re}/M)\, M^2 a_1 (\bar{n}_1\bar{u}_1 + \bar{n}_2\bar{u}_2) \\
&+ (4/15)\sqrt{2/\pi\gamma}\ (\mathrm{Re}/M)\, a_2 a_3 (\bar{n}_1 + \bar{n}_2) = 0 \quad ,
\end{aligned} \tag{19f}$$

where a_1 , a_2 , a_3 are undetermined integration constants.

II. A. 2. Boundary Conditions

For completely diffuse reemission the boundary conditions are quite simple. (See Introduction.):

At $y = + d/2$ (Figure 2) , $u_1 = + (U/2)$; $T_1 = T_I$.

At $y = - d/2$, $u_2 = - (U/2)$, $T_2 = T_{II}$.

Also, $v(\pm d/2) = 0$, but $v \equiv 0$ everywhere [Eqs. (9a) and (9b)], so that Eq. (19a) satisfies this boundary condition automatically.

The sixth boundary condition involves a specification of the density level of the gas between the plates, by choosing ρ or n_1 or n_2 at a given point. Since the results evidently do not depend on the position of this reference point we select

$$n_2 = n_{II} \quad \text{at} \quad y = -d/2 \quad .$$

In non-dimensional form the boundary conditions are as follows:

$$\left.\begin{aligned} \bar{u}_1 &= \tfrac{1}{2} && (20a) \\ \bar{T}_1 &= T_I/T_{II} && (20b) \end{aligned}\right\} \quad \text{at } \bar{y} = \tfrac{1}{2}$$

$$\left.\begin{aligned} \bar{u}_2 &= -\tfrac{1}{2} && (20c) \\ \bar{T}_2 &= 1 && (20d) \\ \bar{n}_2 &= 1 && (20e) \end{aligned}\right\} \quad \text{at } \bar{y} = -\tfrac{1}{2}$$

Plane compressible Couette flow is completely determined by three independent parameters: (Re/M) (or d/λ_{II}) the rarefaction parameter; M^2, the dissipation parameter; and the plate temperature ratio T_I/T_{II} appearing explicitly only in the boundary conditions. The governing equations and boundary conditions [Eqs. (19) and (20)] are all regular in the parameters Re/M , T_I/T_{II} , and M^2 for all finite values of these parameters. In particular, in the limiting case $Re/M \rightarrow 0$ all six equations reduce to algebraic equations, and the six unknown functions approach the (constant) values given by free-molecule flow. In the opposite limiting case $Re/M \rightarrow \infty$, clearly α_1 and α_3 are both

$0(M/Re)$ [Eqs. (19d) and (19e)] . Thus the pairs $\bar{u}_2$ and $\bar{u}_1$, $\bar{n}_2$, and $\bar{n}_1$, and $\bar{T}_2$ and $\bar{T}_1$ each differ by a term of order $M/Re \sim \lambda_{II}/d$, which corresponds to the classical Navier-Stokes regime [Sections II. B and II. C] .

In order to bring out the effect of temperature difference between the two plates we study first the simpler case $M^2 << 1$, and then generalize the technique for obtaining solutions to the case of arbitrary Mach number. All mean flow quantities are easily evaluated once the six functions $\bar{n}_1(\bar{y}) \ldots\ldots \bar{u}_2(\bar{y})$ are determined. For convenience the necessary relations are listed here:

$$\frac{p_{xy}(y)}{\rho_{II}\sqrt{\frac{\mathcal{R}T_{II}}{2\pi}}\;U} = \bar{p}_{xy}(y) = -\bar{n}_1\sqrt{\bar{T}_1}\,(\bar{u}_2 - \bar{u}_1) = -\alpha_1 \tag{21a}$$

$$\frac{P_{xx}(y)}{n_{II}\,k\,T_{II}} = \gamma M^2 (\bar{\rho}\,\bar{u}^2) - \tfrac{1}{2}(\bar{n}_1\bar{T}_1 + \bar{n}_2\bar{T}_2) - \gamma M^2(\bar{n}_1\bar{u}_1^{\,2} + \bar{n}_2\bar{u}_2^{\,2}) \tag{21b}$$

$$\frac{P_{yy}(y)}{n_{II}\,k\,T_{II}} = \frac{P_{zz}(y)}{n_{II}\,k\,T_{II}} = -\tfrac{1}{2}(\bar{n}_1\bar{T}_1 + \bar{n}_2\bar{T}_2) \tag{21c}$$

$$\bar{T}(y) = \frac{\bar{n}_1\bar{T}_1 + \bar{n}_2\bar{T}_2}{\bar{n}_1 + \bar{n}_2} + (\gamma/3)\,M^2\left[\frac{\bar{n}_1(\bar{u}-\bar{u}_1)^2 + \bar{u}_2(\bar{u}-\bar{u}_2)^2}{\bar{n}_1 + \bar{n}_2}\right] \tag{21d}$$

$$\frac{p(y)}{n_{II}\,k\,T_{II}} = \tfrac{1}{2}(\bar{n}_1\bar{T}_1 + \bar{n}_2\bar{T}_2) + (\gamma/6)\,M^2\left[\bar{n}_1(\bar{u}-\bar{u}_1)^2 + \bar{n}_2(\bar{u}-\bar{u}_2)^2\right] \tag{21e}$$

$$\frac{q_y}{n_{II}\,k\,T_{II}\,\sqrt{(2/\pi)\mathcal{R}T_{II}}} = (\gamma M^2/2)\,\bar{p}_{xy}\,\bar{u} + \bar{n}_2\bar{T}_2^{\,3/2}\left[1 + (\gamma M^2/4)(\bar{u}_2^{\,2}/\bar{T}_2)\right] - \bar{n}_1\bar{T}_1^{\,3/2}\left[1 + \frac{\gamma M^2}{4}\,\frac{\bar{u}_1^{\,2}}{\bar{T}_1}\right]. \tag{21f}$$

The expressions for $\rho(y)$ and $u(y)$ have already been given [Eqs. (17) and 18)] .

According to Eqs. (21b), (21c), and (21e), $P_{yy} \rightarrow P_{xx} \rightarrow -p$ when $M^2 \rightarrow 0$, <u>or</u> when $Re/M \rightarrow \infty$. In general however, $P_{yy} \neq P_{xx} \neq -p$, so that $p_{ii} \neq 0$, in spite of the fact that $\operatorname{div} \vec{u} \equiv 0$. This behavior shows again the inadequacy of any concept relating the stresses to the purely local mean velocity gradients in a rarefied gas flow.

II. B. Low Mach Number Flow with Arbitrary Plate Temperature Ratio

In this case ($M^2 << 1$) the basic equations [Eqs. (19)] are considerably simplified:

<u>Continuity</u>

$$\bar{n}_1 \sqrt{\bar{T}_1} = \bar{n}_2 \sqrt{\bar{T}_2} \tag{22a}$$

<u>Momentum</u>

$$\bar{n}_1 \sqrt{\bar{T}_1} \quad (\bar{u}_2 - \bar{u}_1) = \alpha_1 \tag{22b}$$

$$\bar{n}_1 \bar{T}_1 + \bar{n}_2 \bar{T}_2 = \alpha_2 \tag{22c}$$

<u>Energy</u>

$$\bar{n}_1 \sqrt{\bar{T}_1} \quad (\bar{T}_2 - \bar{T}_1) = \alpha_2 \alpha_3 \tag{22d}$$

<u>Stress</u>

$$(d/d\bar{y}) (\bar{n}_1 \bar{u}_1 \bar{T}_1 + \bar{n}_2 \bar{u}_2 \bar{T}_2) + \frac{1}{\sqrt{2\pi\gamma}} (Re/M) \alpha_1 (\bar{n}_1 + \bar{n}_2) = 0 \tag{22e}$$

<u>Heat Flux</u>

$$(d/d\bar{y})(\bar{n}_1 \bar{T}_1{}^2 + \bar{n}_2 \bar{T}_2{}^2) + (4/15) \sqrt{(2/\pi\gamma)} (Re/M) \alpha_2 \alpha_3 (\bar{n}_1 + \bar{n}_2) = 0 \tag{22f}$$

Of course the boundary conditions [Eqs. (20)] are unchanged.

As expected, the small Mach number simplification leads to a split in the system of equations; namely, Eqs. (22 a, c, d, f) govern the four functions $\overline{n}_1$, $\overline{n}_2$, $\overline{T}_1$, and $\overline{T}_2$, while Eqs. (22 b, e) describe the behavior of $\overline{u}_1$ and $\overline{u}_2$. This independence of thermodynamic variables and dynamic variables is a basic feature of low Mach number flow (10). The heat flux, temperature and density profiles so obtained are clearly valid for the problem of convective heat transfer between two stationary plates.

The three algebraic equations for the four functions $\overline{n}_1$, $\overline{n}_2$, $\overline{T}_1$, and $\overline{T}_2$ [Eqs. (22a), (22c) , and (22d)] permit these variables to be eliminated in favor of a single function $G(\overline{y})$; then Eq. (22f) furnishes an ordinary first-order non-linear equation for $G(\overline{y})$. It turns out to be most convenient to take $\overline{n}_1 \sqrt{\overline{T}_1} = \overline{n}_2 \sqrt{\overline{T}_2} = [G(\overline{y})]^{-1}$. Then Eq. (22c) is reduced to

$$\sqrt{\overline{T}_1} + \sqrt{\overline{T}_2} = \alpha_2 G \tag{23a}$$

while Eq. (22d) becomes

$$\overline{T}_2 - \overline{T}_1 = \alpha_2 \alpha_3 G \quad . \tag{23b}$$

From Eqs.(23a) and (23b) one finds

$$\overline{T}_1 (\overline{y}) = (1/4) (\alpha_2 G - \alpha_3)^2 \tag{24a}$$

$$\overline{T}_2 (\overline{y}) = (1/4) (\alpha_2 G + \alpha_3)^2 \tag{24b}$$

so that

$$\overline{n}_1 (\overline{y}) = \frac{2}{G (\alpha_2 G - \alpha_3)} \tag{24c}$$

$$\overline{n}_2 (\overline{y}) = \frac{2}{G (\alpha_2 G + \alpha_3)} \quad . \tag{24d}$$

After substitution, Eq. (22f) becomes

$$(\alpha_2^{\,2}\, G^2 - \alpha_3^{\,2})\,(d/d\bar{y})\,(\alpha_2^{\,2}\, G^2) + (64/15)\,\sqrt{(2/\pi\gamma)}\,(\mathrm{Re}/M)\,\alpha_2\,\alpha_3 = 0\,. \tag{25a}$$

Integration of this equation yields

$$G(\bar{y}) = \left[(\alpha_3/\alpha_2)^2 \pm (2/\alpha_2)\,(\alpha_4 - \frac{32}{15}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\alpha_3}{\alpha_2}\,\frac{\mathrm{Re}}{M}\,\bar{y}\,)^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad , \tag{25b}$$

where α_4 is the new integration constant. The plus sign in $G(\bar{y})$ is taken owing to the fact that

$$\alpha_2^{\,2}\, G^2 - \alpha_3^{\,2} = (2\alpha_2/\bar{n}) \quad ,$$

and

$$\alpha_2 > 0\,, \quad \bar{n} > 0 \qquad \text{always.}$$

By using Eqs. (24), one finds that the boundary conditions (Eqs. 20 b, d, and e) lead to the following conditions on $G(\bar{y})$:

$$G = \frac{2L + \alpha_3}{\alpha_2} \qquad \text{at} \quad \bar{y} = \tfrac{1}{2} \tag{26a}$$

$$\left.\begin{aligned} G &= 1 \qquad &(26b)\\ \alpha_2 + \alpha_3 &= 2 \qquad &(26c)\end{aligned}\right\}\ \text{at}\ \bar{y} = -\tfrac{1}{2}$$

$\left[\text{Here } L = \sqrt{(T_I/T_{II})}\right]$. These three conditions are sufficient for the evaluation of α_2, α_3, and α_4; the results are

$$\alpha_2 = \frac{\left[(1+\beta)(L^4 + 2L^3 + \beta) + L^2\right]^{\frac{1}{2}} - (L^3 + 2\,L^2 - \beta - 1)}{(1 + \beta - L^2)} \tag{27a}$$

$$\alpha_3 = 2 - \alpha_2 = \frac{-\left[(1+\beta)(L^4 + 2L^3 + \beta) + L^2\right]^{\frac{1}{2}} + (L^3 + \beta + 1)}{(1 + \beta - L^2)} \tag{27b}$$

$$\alpha_4 = 1 - 2\,(1+\beta)\,(\alpha_3/\alpha_2) + (\alpha_3/\alpha_2)^2 \quad , \tag{27c}$$

in which the abbreviation $\beta = (8/15)\sqrt{2/\pi\gamma}\,(Re/M)$ is employed.

Once $G(\bar{y})$ is known, $\bar{n}_1$, $\bar{n}_2$, $\bar{T}_1$, $\bar{T}_2$ are completely determined, and so are the average density ρ, temperature T, pressure p and the heat flux q_y. We obtain explicit solutions for these quantities, as follows:

$$\rho/\rho_{II} = (a_4 - 4\beta \frac{a_3}{a_2} \frac{y}{d})^{-\frac{1}{2}} \tag{28a}$$

$$T/T_{II} = (a_2/2)\,(a_4 - 4\beta \frac{a_3}{a_2} \frac{y}{d})^{\frac{1}{2}} \tag{28b}$$

$$p/p_{II} = (a_2/2) \tag{28c}$$

$$\frac{q_y}{\left(\frac{k_{cII} T_{II}}{d}\right)} = (4/15)\sqrt{(2/\pi\gamma)}\; a_2\, a_3\, (Re/M) \quad , \tag{28d}$$

where the subscript II denotes quantities evaluated at T_{II}, ρ_{II}.

By introducing the Stanton number

$$C_H = \frac{q_y}{\rho_{II}\, C_{p_{II}}\, U\,(T_{II} - T_I)}$$

and using Eq. (28d), we obtain

$$C_H\, M = (4/15)\sqrt{(2/\pi\gamma)}\; \frac{a_2\, a_3}{Pr\,(1 - L^2)} \quad . \tag{29}$$

Once the solutions for $\bar{n}_1$, $\bar{n}_2$, $\bar{T}_1$, and $\bar{T}_2$ are obtained one can solve for $\bar{u}_1$ and $\bar{u}_2$ from Eqs. (22b) and (22e). But one is interested in the average velocity $\bar{u}$ rather than $\bar{u}_1$ or $\bar{u}_2$. According to Eq. (24)

$$\bar{n}_1\, \bar{u}_1\, \bar{T}_1 + \bar{n}_2\, \bar{u}_2\, \bar{T}_2 = (a_2/2)\,(\bar{u}_1 + \bar{u}_2) + \frac{a_3\, a_1}{2}$$

while

$$\bar{u}\,(\bar{y}) = \frac{\bar{n}_1\bar{u}_1 + \bar{n}_2\bar{u}_2}{\bar{n}_1 + \bar{n}_2} = \frac{\bar{u}_1 + \bar{u}_2}{2} - \frac{a_3\, a_1}{2\, a_2} \quad .$$

Therefore,

$$d/d\bar{y}\ (\bar{n}_1\bar{u}_1\bar{T}_1 + \bar{n}_2\bar{u}_2\bar{T}_2) = a_2\,(d\bar{u}/d\bar{y}) \quad , \quad * \tag{30}$$

and Eq. (22e) is readily integrated to give

$$\bar{u} = (15/16)\,(a_1/a_3)\,\left(a_4 - 4\beta\,\frac{a_3}{a_2}\,\bar{y}\right)^{\frac{1}{2}} + a_5 \tag{31}$$

where a_1 and a_5 are undetermined integration constants. The boundary conditions for $\bar{u}_1$ and $\bar{u}_2$ [Eqs. (20a, c)] can also be converted into two boundary conditions for $\bar{u}$:

$$\bar{u} = \tfrac{1}{2} + L\,(a_1/a_2) \qquad \text{at } \bar{y} = +\tfrac{1}{2}$$

$$\bar{u} = -\tfrac{1}{2} - (a_1/2)\,\left(1 + \frac{a_3}{a_2}\right) = -\tfrac{1}{2} - (a_1/a_2) \qquad \text{at } \bar{y} = -\tfrac{1}{2} \quad ,$$

with a_2 , a_3 , and a_4 given by Eqs. (27)

These two conditions lead to the evaluation of a_1 and a_5 in terms of the known quantities a_2 , a_3 , and a_4 .

$$a_1 = \frac{1}{(15/16)(1/a_3)\left[\left(a_4 - 2\beta\,\frac{a_3}{a_2}\right)^{\frac{1}{2}} - \left(a_4 + 2\beta\,\frac{a_3}{a_2}\right)^{\frac{1}{2}}\right] - \dfrac{1+L}{a_2}}$$

$$a_5 = -\tfrac{1}{2}\cdot\frac{(15/16)(1/a_3)\left[\left(a_4 - 2\beta\,\frac{a_3}{a_2}\right)^{\frac{1}{2}} + \left(a_4 + 2\beta\,\frac{a_3}{a_2}\right)^{\frac{1}{2}}\right] + \dfrac{1-L}{a_2}}{(15/16)(1/a_3)\left[\left(a_4 - 2\beta\,\frac{a_3}{a_2}\right)^{\frac{1}{2}} - \left(a_4 + 2\beta\,\frac{a_3}{a_2}\right)^{\frac{1}{2}}\right] - \dfrac{1+L}{a_2}} \quad ,$$

---

* This result means that $p_{xy} = \mu_c\,(du/dy)$ in the limiting case $M^2 << 1$ regardless of the plate temperature ratio.

Therefore,

$$u/U = \frac{(15/16)\,(1/a_3)}{(15/16)(1/a_3)\left[(a_4 - 2\beta\,\frac{a_3}{a_2})^{\frac{1}{2}} - (a_4 + 2\beta\,\frac{a_3}{a_2})^{\frac{1}{2}}\right] - \frac{1+L}{a_2}}$$

$$\times \left\{(a_4 - 4\beta\,\frac{a_3}{a_2}\,\frac{y}{d})^{\frac{1}{2}} - \frac{1}{2}\left[(a_4 - 2\beta\,\frac{a_3}{a_2})^{\frac{1}{2}} + (a_4 + 2\beta\,\frac{a_3}{a_2})^{\frac{1}{2}}\right] + - \frac{1-L}{2a_2}\right\} \qquad (32)$$

By employing Eq. (21a), we can express the shearing stress as

$$\frac{p_{xy}}{\rho_{II}\sqrt{\gamma \mathcal{R} T_{II}}\;\; U} = - \frac{a_1}{\sqrt{2\pi\gamma}}$$

where $a_1 < 0$.

Define the skin friction coefficient C_D

$$C_D = \frac{p_{xy}}{\frac{1}{2}\,\rho_{II}\,U^2}$$

then

$$C_D M = -(C_D M)_o\, a_1$$

where $(C_D M)_o$ = the value of $C_D M$ for free molecular flow ($Re/M = 0$) = $\sqrt{(2/\pi\gamma)}$.

In this limiting case of low Mach number flow $P_{yy} \cong -p$ and $p_{yy} \cong 0$ [Eqs. (21c, e)] so that both P_{yy} and p are constant across the flow, as shown by Eq. (28c). Also the energy equation [Eq. (9e)] states that q_y = constant in the absence of dissipation. When in addition $L^2 = (T_I/T_{II}) \longrightarrow 1$, then $a_3 = 0\ \left(\frac{T_I - T_{II}}{T_I}\right)$, as shown by Eq. (27b) and all of these results [Eqs. (28a), (28b), (29), and (31)] reduce to the solutions found previously in Reference 10.

As a numerical example we take $T_I/T_{II} = 4$; the velocity and temperature profiles for this case are shown in Figures 3 and 4. In Figures 5 and 6 the normalized skin-friction and heat transfer coefficients are plotted against Re/M. These results are discussed in Section III.

II. C. Arbitrary Mach Number and Plate Temperature Ratio

When the Mach number and plate temperature ratio are both arbitrary the velocity and temperature fields are closely coupled. In that case the four algebraic conservation relations [Eqs. (19a) - (19d)] allow us to replace the six unknown functions by two independent functions $F(\bar{y})$ and $G(\bar{y}) = (n_1 \sqrt{T_1})^{-1} = (n_2 \sqrt{T_2})^{-1}$, and Eqs. (19e) and (19f) furnish two simultaneous, first-order, non-linear ordinary differential equations for F and G.

In the expressions for $\bar{n}_1$, $\bar{T}_1$, $\bar{n}_2$, $\bar{T}_2$ given in Section II. B the constant α_3 is replaced by the function F(y). Thus, Eqs. (19a) and (19c) are automatically satisfied by taking

$$\bar{n}_1(\bar{y}) = \frac{2}{G(\alpha_2 G - F)} \tag{33a}$$

$$\bar{n}_2(\bar{y}) = \frac{2}{G(\alpha_2 G + F)} \tag{33b}$$

$$\bar{T}_1(\bar{y}) = (1/4)(\alpha_2 G - F)^2 \tag{33c}$$

$$\bar{T}_2(\bar{y}) = (1/4)(\alpha_2 G + F)^2 \tag{33d}$$

and Eqs. (19b) and (19d) yield

$$\bar{u}_1(\bar{y}) = \tfrac{1}{2}\left[\frac{4}{\gamma M^2}\cdot\frac{\alpha_2}{\alpha_1}(\alpha_3 - F) - \alpha_1 G\right] \tag{33e}$$

$$\bar{u}_2(\bar{y}) = \tfrac{1}{2}\left[\frac{4}{\gamma M^2}\cdot\frac{\alpha_2}{\alpha_1}(\alpha_3 - F) + \alpha_1 G\right] \tag{33f}$$

After substitution, Eqs. (19e) and (19f) become two integrable equations governing $G(\overline{y})$ and $F(\overline{y})$, as follows:

$$(a_2^2 G^2 - F^2)(dF/d\overline{y}) - \frac{8}{\sqrt{2\pi\gamma}} (Re/M) \frac{a_1}{\left(\frac{4}{\gamma M^2}\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)} = 0 \quad (34)$$

$$G(dG/d\overline{y}) + \lambda_1 F(dF/d\overline{y}) = 0 \quad (35)$$

where

$$\lambda_1 = \lambda_1(a_1, a_2, \gamma M^2) = (1/3) \cdot \frac{21 + \gamma M^2 \left(\frac{a_1}{a_2}\right)^2 + \frac{32}{\gamma M^2}\left(\frac{a_2}{a_1}\right)^2}{\gamma M^2 a_1^2 + 5 a_2^2} \quad . \quad (35a)$$

Eq. (35) immediately yields the relation

$$G^2 + \lambda_1 F^2 = a_5 \quad . \quad (36)$$

By employing this expression, Eq. (34) can also be integrated. Thus

$$\lambda_3 F^3 - \lambda_2 a_5 F + \frac{8}{\sqrt{2\pi\gamma}} \cdot (Re/M) \cdot (a_1/a_2)\,\overline{y} + a_4 = 0 \quad , \quad (37)$$

where a_4, a_5 are two new integration constants, and

$$\lambda_2 = \lambda_2(a_1, a_2, \gamma M^2) = a_2\left(\frac{4}{\gamma M^2} \cdot \frac{a_2}{a_1} - \frac{a_1}{a_2}\right)$$

$$\lambda_3 = \lambda_3(a_1, a_2, \gamma M^2) = (1/3)\left(a_2 \lambda_1 + \frac{1}{a_2}\right)\left(\frac{4}{\gamma M^2}\frac{a_2}{a_1} - \frac{a_1}{a_2}\right) .$$

When the five boundary conditions [Eq. (20)] are converted into conditions on G(y) and F(y) by employing Eqs. (33), one obtains

$$\text{at } \bar{y} = -\tfrac{1}{2} \quad \begin{cases} G = 1 & \text{(38a)} \\ F = 2 - a_2 & \text{(38b)} \\ (a_2/a_1)(a_3 - F) + \dfrac{\gamma M^2}{4} a_1 G + \dfrac{\gamma M^2}{4} = 0 & \text{(38c)} \end{cases}$$

$$\text{at } \bar{y} = +\tfrac{1}{2} \quad \begin{cases} G = -\dfrac{\dfrac{\gamma M^2}{4} a_1 - 2 L a_2 - a_2 a_3}{(\gamma M^2/4) a_1^2 + a_2^2} & \text{(38d)} \\ F = -\dfrac{(\gamma M^2/2) L a_1^2 + (\gamma M^2/4) a_1 a_2 - a_2^2 a_3}{(\gamma M^2/4) a_1^2 + a_2^2} & \text{(38e)} \end{cases}$$

By substituting these conditions into Eqs. (36) and (37) one gets a system of five algebraic equations governing the five a's:

$$\left(\frac{\frac{\gamma M^2}{4} a_1 - 2 L a_2 - a_2 a_3}{(\gamma M^2/4) a_1^2 + a_2^2} \right)^2 + \lambda_1 \left[\left(\frac{\frac{\gamma M^2}{2} L a_1^2 + \frac{\gamma M^2}{4} a_1 a_2 - a_2^2 a_3}{(\gamma M^2/4) a_1^2 + a_2^2} \right)^2 - (2 - a_2)^2 \right] - 1 = 0 \tag{39a}$$

$$\lambda_3 \left[\left(\frac{\frac{\gamma M^2}{2} L a_1 + \frac{\gamma M^2}{4} a_1 a_2 - a_2^2 a_3}{(\gamma M^2/4) a_1^2 + a_2^2} \right)^3 + (2 - a_2)^3 \right]$$
$$- \lambda_2 \left[1 + \lambda_2 (2 - a_2)^2 \right] \tag{39b}$$
$$\times \left[\left(\frac{\frac{\gamma M^2}{2} L a_1 + \frac{\gamma M^2}{4} a_1 a_2 - a_2^2 a_3}{(\gamma M^2/4) a_1^2 + a_2^2} \right) + (2 - a) \right] - \frac{8}{\sqrt{2\pi\gamma}} \frac{Re}{M} \frac{a_1}{a_2} = 0$$

$$a_3 = 2 - a_2 - (\gamma M^2/4)(a_1/a_2)(1 + a_1) \tag{39c}$$

$$\alpha_4 = (\lambda_1 \lambda_2 - \lambda_3)(2 - \alpha_2)^3 + \lambda_2 (2 - \alpha_2) + (4/\sqrt{2\pi\gamma})\,(\mathrm{Re}/M)\,(\alpha_1/\alpha_2) \quad (39d)$$

$$\alpha_5 = 1 + \lambda_1 (2 - \alpha_2)^2 \quad (39e)$$

Solving numerically for the α's is not so tedious as it seems. The first two equations, in which the value of α_3 is given by Eq. (39c), can be solved simultaneously for α_1 and α_2 by means of trial and error. The fact that $(-\alpha_1)$ always varies monotonically between unity and zero as Re/M increases simplifies the iteration procedure considerably. By consulting the $C_D M/(C_D M)_o$ diagram, one can make a fairly good first estimate of α_1 , then one can solve for α_2 from Eq. (39a), and substitute these values of α_1 and α_2 into Eqs. (39b) to check if the Re/M so obtained deviates from the given value. This procedure converges very rapidly to the final result. As soon as α_1 , α_2 are known, the other three constants are readily determined from Eqs. (39 c, d, and e).

All mean quantities of interest are expressed in terms of F, G, and the five α's as follows:

$$u/U = (2/\gamma M^2)\left[(\alpha_3\alpha_2/\alpha_1) - \left(\frac{\alpha_2}{\alpha_1} + \frac{\gamma M^2}{4}\,\frac{\alpha_1}{\alpha_2}\right) F\right] \quad (40a)$$

$$T/T_{II} = 1/4\left[1 + (\gamma M^2/3)\,(\alpha_1/\alpha_2)^2\right](\alpha_2^{\,2} G^2 - F^2) \quad (40b)$$

$$\rho/\rho_{II} = \frac{2\alpha_2}{\alpha_2^{\,2} G^2 - F^2} \quad (40c)$$

$$\frac{P_{xy}}{\rho_{II}\sqrt{\gamma R\, T_{II}}\;\; U} = -\,(\alpha_1 / \sqrt{2\pi\gamma}) \quad (40d)$$

$$\frac{q_y}{(k_{cII}T_{II}/d)} = (4/15)\sqrt{(2/\pi\gamma)}\ (Re/M)\ \alpha_1 \left(\frac{\alpha_2}{\alpha_1} + \frac{\gamma M^2}{4}\frac{\alpha_1}{\alpha_2} F\right). \quad (40e)$$

Hence,

$$C_D M/(C_D M)_o = -\alpha_1 \quad \text{where} \quad (C_D M)_o = \sqrt{(2/\pi\gamma)} \quad (40f)$$

and

$$C_H M = (4/15)\sqrt{(2/\pi\gamma)}\ \frac{\alpha_1}{Pr\,(1 - L^2)} \left(\frac{\alpha_2}{\alpha_1} + \frac{\gamma M^2}{4}\frac{\alpha_1}{\alpha_2} F\right). \quad (40g)$$

In practice, for a particular $\bar{y}$, more than one value of F is obtained, because Eq. (37) is a third order algebraic equation always having three real roots. A typical variation of $F(\bar{y})$ is sketched in Figure 7. Now $F(\bar{y})$ is a continuous function of $\bar{y}$ in the gas, so that only one of the three possible branches is physically acceptable. When $L^2 = 1$, the velocity profile is always anti-symmetrical about the mid-plane, so that $F(\bar{y})$ is also anti-symmetrical [Eq. (40a)] . Thus, only the intermediate heavily-lined branch (2) is physically realistic in this case. When $L^2 > 1$ the behavior of $F(\bar{y})$ is more complicated. According to Eq. (34) for $(dF/d\bar{y})$, when $Re/M \longrightarrow \infty$, $F(\bar{y}) \longrightarrow$ constant. As Re/M decreases the curve of F vs. $\bar{y}$ gradually rotates in the counterclockwise direction at first, and the pivot point moves smoothly along the F- axis in one direction. At some intermediate value of Re/M the curve of F vs. $\bar{y}$ reverses its direction of rotation and finally reaches a horizontal position in the limit $Re/M \longrightarrow 0$. Evidently only the heavily-lined branch is physically realistic for any arbitrary values of L^2, M^2, and Re/M. [See Section III. E.]

As a numerical illustration we take $M = 3$, $L^2 = T_{II}/T_I = 4$. The skin-friction and heat transfer coefficients are shown as functions of Re/M in Figures 5 and 6, and the velocity and temperature profiles are plotted in Figures 8 and 9. These results are discussed in Section III.

II. D. Special Case of Equal Plate Temperatures, But Arbitrary Mach Number

Although this special case is included in the general analysis given in Section II. C, there are certain important simplifications. When both plates have the same temperature, then obviously $\bar{u}_1$, $\bar{u}_2$, and $\bar{u}$ are all anti-symmetrical, while $\bar{T}_1$, $\bar{T}_2$, and $\bar{T}$ (or $\bar{n}$) must be symmetrical with respect to $\bar{y} = 0$, i. e.,

$$\bar{u}_1 (\bar{y}) = - \bar{u}_2 (-\bar{y}) \quad \text{and} \quad \bar{T}_1 (\bar{y}) = \bar{T}_2 (-\bar{y}) \quad .$$

From Eqs. (33a) and (40a), one sees that

$$F (\bar{y}) = - F (-\bar{y}) \tag{41a}$$

$$G (\bar{y}) = G (-\bar{y}) \quad . \tag{41b}$$

Therefore, $F(0) = 0$, and from Eq. (37), we have $a_4 = 0$. At $\bar{y} = 0$, one can conclude from Eqs. (41b), (33e), and (33f) that

$$a_3 = 0 \quad ,$$

since a_1 and a_2 remain finite for all possible values of M and Re/M .

The non-trivial a's are then governed by Eqs. (39c , d, and e)

$$a_2 (2 - a_2) - (\gamma M^2/4)\, a_1 (1 + a_1) = 0 \tag{42a}$$

$$(\lambda_1 \lambda_2 - \lambda_3)(2 - a_2)^3 + \lambda_2 (2 - a_2) + \frac{4}{\sqrt{2\pi\gamma}} (Re/M)(a_1/a_2) = 0 \tag{42b}$$

$$a_5 = 1 + \lambda_1 (2 - a_2)^2 \tag{42c}$$

So one can easily express a_2 in terms of a_1 by means of Eq. (42a). Selecting the positive root,

$$a_2 = 1 + \sqrt{1 - (\gamma M^2/4)\, a_1 (1 + a_1)} \quad .$$

By substituting this expression for a_2 into Eq. (42b), one solves for a_1, and then one obtains a_5 from Eq. (42c).

The variation of skin-friction coefficient with the rarefaction parameter Re/M for the case M = 3, $T_I/T_{II} = 1$ is shown in Figure 5, and the velocity and temperature profiles are plotted in Figures 10 and 11.

III. DISCUSSION AND CONCLUSIONS

III. A. Skin-Friction and Heat Transfer Coefficients

As expected from the structure of the basic differential equations [Eq. (19)] the variation of the skin friction and heat transfer coefficients with Re/M is smooth and continuous over the whole range (Figures 5 and 6). For all values of Mach number and plate temperature ratio $C_D M$ approaches the value given by the solution of the Navier-Stokes equations as $Re/M \longrightarrow \infty$ (Appendix), and $C_D M$ approaches the free molecule flow value of $\sqrt{(\pi\gamma/2)}$ as $Re/M \longrightarrow 0$. According to Eq. (A-9),

$$(C_D M)_{\substack{\text{Navier}\\ \text{Stokes}}} = \frac{2}{(Re/M)}\left[\frac{1}{2}\left\{1 + (T_I/T_{II}) + \left((\gamma-1)/6\right) \Pr M^2\right\}\right] . \qquad \text{(A-9)}$$

This behavior suggests that the rarefaction parameter should be renormalized by replacing Re/M, based on physical quantities evaluated at T_{II}, ρ_{II}, with a new parameter $(Re/M)^*$, where

$$(Re/M)^* = (Re/M) \cdot \frac{2}{1 + (T_I/T_{II}) + \left((\gamma-1)/6\right) \Pr M^2} .$$

This procedure amounts to evaluating the "proper" mean free path at a density corresponding to a certain "kinetic temperature", i.e.,

$$(\lambda/d)^* = (\lambda_{II}/d) \cdot (T_K/T_{II}) ,$$

where T_K/T_{II} is given by the bracket in Eq. (A-9). Evidently $(\lambda/d)^* >> (\lambda_{II}/d)$ for high values of M^2 and/or T_I/T_{II}.

In Figure 12 the drag coefficient is replotted in terms of this new rarefaction parameter $(Re/M)^*$. In these coordinates all the curves deviate only slightly from the "basic" curve corresponding to $M^2 = 0$, $T_I/T_{II} = 1$. Appreciable deviations ($\cong$ 10 per cent) from the classical

Navier-Stokes solution occur even at values of Re/M as high as 30, or $(\lambda/d)^* = 1/20$. The approach to free-molecule flow is also quite slow, because the solutions are simple algebraic functions of Re/M. On the other hand, the major portion of the transition from the classical Navier-Stokes regime to the highly rarefied regime occurs over an interval of less than a decade in Re/M or gas density $\left[3 < (Re/M)^* < 30, \text{ or } \tfrac{1}{2} < (\lambda/d)^* < 1/20\right]$. This behavior must be closely connected with the "cascading" effect of particle collisions in the gas. When a particle suffers only one or two collisions in passing from one plate to the other the effect on the particle velocity distribution is small. But when 5 - 10 collisions occur, especially with particles emitted from the opposite plate, the effect is cumulative, and almost all the particles quickly forget their original place of birth.

Since the Mach number appears in the definition of "kinetic temperature" only as the factor $((\gamma-1)/6)$ Pr $M^2 = (2/27)\,M^2$ for a monatomic gas, the drag coefficient is rather insensitive to Mach number for $M < 1.5$. This conclusion agrees with the experimental results of Bowyer and Talbot (3), Kuhlthau (9), and Chiang (4) for cylindrical Couette flow with small ratio of annulus width to cylinder radius.

III. B. Shear Stress and Normal Heat Flux

By utilizing Eq. (21a), one finds that

$$\frac{p_{xy}}{\mu_c\,(du/dy)} = -\,(Re/M)\,\frac{\alpha_1}{\sqrt{2\pi\gamma}}\left(\frac{1}{\bar{n}}\,\frac{d\bar{u}}{d\bar{y}}\right)^{-1} . \tag{43}$$

But according to Eq. (40a),

$$(d\bar{u}/dy) = -\,\tfrac{1}{2}\left[\frac{4}{\gamma M^2}\,\frac{\alpha_2}{\alpha_1} + \frac{\alpha_1}{\alpha_2}\right]\frac{dF}{d\bar{y}} ,$$

so that the ratio $\dfrac{p_{xy}}{\mu_c\,(du/dy)}$ is constant across the flow $\left[\text{Eq. (43)}\right]$, and is given by

$$\frac{p_{xy}}{\mu_c\,(du/dy)} = (\tilde{\mu}/\mu_c) = \frac{1}{\left[1+\frac{\gamma M^2}{3}(a_1/a_2)^2\right]}\;\frac{\left[\frac{4}{\gamma M^2}(a_2/a_1) - (a_1/a_2)\right]}{\left(\frac{4}{\gamma M^2}(a_2/a_1)+(a_1/a_2)\right)}. \quad (44)$$

Clearly $\tilde{\mu} \to \mu_c$ when $M^2 \to 0$, or when $Re/M >> 1$, for any values of M^2 and T_I/T_{II}. On the other hand at any finite fixed values of T_I/T_{II} and Re/M the ratio $\tilde{\mu}/\mu_c$ decreases rapidly with increasing plate Mach number. This behavior is connected with the fact that the(non-dimensional) shear stress is not much affected by plate velocity, but the gas temperature is everywhere large in a rarefied gas at high plate Mach number. In Figure 13 this behavior is shown schematically for the limiting case $Re/M = 0$. In Figure 15 the variation of $\tilde{\mu}/\mu_c$ with Re/M is illustrated for two values of T_I/T_{II} at $M = 3$. Again one sees that the transition from the classical Navier-Stokes regime to the almost free molecule flow regime occurs over about one decade in the rarefaction parameter.

We observe that $\tilde{\mu}/\mu_c \to 1$ when $T_I/T_{II} >> 1$ (Figure 13). In this case the condition of zero normal velocity $\left[\text{Eq. (19a)}\right]$ shows that $n_{II}/n_I = \sqrt{T_I/T_{II}} >> 1$ when $Re/M = 0$, and the mean temperature in the gas approaches the geometric mean value $\sqrt{T_I\,T_{II}}$, regardless of the Mach number (Eq. (21d). Thus the "proper" Mach number is based on this mean temperature, or

$$\tilde{M}^2 = M^2 \sqrt{T_{II}/T_I} \to 0 \quad \text{when} \quad T_I/T_{II} >> 1 \,,$$

and it is not surprising that $p_{xy} \to \mu_c\,(du/dy)$ in this limiting case. In fact this argument is valid for all values of Re/M and M^2. To prove this conclusion formally, one observes that

$$\frac{P_{yy}}{n_{II}\, k\, T_{II}} = -(a_2/2) = -\tfrac{1}{2}(\bar{n}_1 \bar{T}_1 + \bar{n}_2 \bar{T}_2) \cong -\tfrac{1}{2}\bar{n}_2 \bar{T}_2 \sqrt{\bar{T}_1/\bar{T}_2} \quad ,$$

when $T_I/T_{II} >> 1$, so that $a_2 >> 1$. Then Eq. (44) shows that $\tilde{\mu}_c \rightarrow \mu_c$.

Similar remarks apply to the ratio of normal heat flux q_y to $-k_c(dT/dy)$. One finds that

$$\frac{q_y}{-k_c\,(dT/dy)} = \frac{(2/15)\,\gamma M^2\left[\left(\frac{4}{\gamma M^2}\,(a_2/a_1)+(a_1/a_2)\right)\left(\frac{4}{\gamma M^2}(a_2/a_1)-(a_1/a_2)\right)\right]}{\left[1+\frac{\gamma M^2}{3}\,(a_1/a_2)^2\right]\left[\;1+a_2^{\;2}\;\lambda_1\right]} \quad .$$

The variation of this ratio with the parameters T_I/T_{II} and M^2 in the limiting case $Re/M \rightarrow 0$ is shown in Figure 14.

III. C. Mean Temperature and Mean Velocity Profiles

When $M^2 << 1$ the mean gas temperature approaches the geometric mean $\sqrt{T_I T_{II}}$ in the limit $Re/M \rightarrow 0$, as expected from the statistical weighting of the two Maxwellian streams. For arbitrary Mach numbers the gas temperature in this limiting case is equal to the geometric mean, plus a "kinetic" term [Eq. (21d) and Figure 9] . The temperature profiles pass smoothly from this free-molecule flow behavior to the behavior predicted by the Navier-Stokes-Fourier relations as Re/M increases.

The behavior of the mean velocity profiles is more interesting. By starting with Eq. (21a) for the shear stress, and introducing the boundary conditions [Eq. (20)] , one can derive a very simple relation for the ratio of the velocity slip at the two plates. Thus,

$$\text{at } y = + d/2 , \qquad \bar{n}_1 \sqrt{\bar{T}_1} \; (\tfrac{1}{2} - \bar{u}_2) = - a_1 \tag{45a}$$

$$\text{at } y = - d/2 , \qquad (\tfrac{1}{2} + \bar{u}_1) = - a_1 \quad . \tag{45b}$$

By utilizing the expression for mean velocity [Eq. (18)] at $y = \pm d/2$, the quantities $\bar{u}_2 (+ d/2)$ and $\bar{u}_1 (- d/2)$ are eliminated in favor of $\bar{u}$, and Eqs. (45a) and (45b) become

$$(\bar{n}_1/\bar{n}_2) \; \bar{n} \; \sqrt{\bar{T}_1} \; \left[\tfrac{1}{2} - \bar{u}\,(d/2) \right] = - a_1 \tag{45c}$$

$$(\bar{n}/\bar{n}_1) \left[\bar{u}\,(- d/2) + \tfrac{1}{2} \right] = - a_1 \quad . \tag{45d}$$

But Eqs. (19a) and (19c) yield the relation

$$\bar{n} \sqrt{\bar{T}_1 \, \bar{T}_2} = \text{constant} = a_2 \; . \tag{45e}$$

By utilizing this last relation, Eq. (19a), and the boundary conditions on $\bar{T}_1$ and $\bar{T}_2$ one obtains

$$\frac{\tfrac{1}{2} - \bar{u}\,(+ d/2)}{\bar{u}\,(- d/2) + \tfrac{1}{2}} = \sqrt{T_I/T_{II}} \quad , \tag{46}$$

<u>independently of Re/M or M^2</u> .

In the Navier-Stokes regime most of the gas follows the hot plate (Figures 3 and 8) because $\mu_c \sim T$, and the stopping power of the hot plate is larger. However, the situation is reversed as the gas density decreases, because according to Eq. (46) the velocity "slip" at the hot plate is larger than at the cold plate. Finally, in the limit $Re/M \to 0$, most of the gas follows the cold plate.

When $T_I/T_{II} >> 1$, the velocity slip at the cold plate is small, because the number density of particles emitted from the cold plate is much larger than the number density emitted from the hot plate. In this case $p_{xy} \rightarrow \mu_c$ (du/dy) (Section III. B), yet the flow bears no resemblance to the predictions of the classical Navier-Stokes equations with no slip. Especially in the highly rarefied flow regime the mean velocity is determined by the statistical weighting of the influence of the two plates.

III. D. Comparison of Present Results with Maxwell's Velocity Slip Relation

When the gas is not too rarefied Maxwell suggested that the Newtonian relation $p_{xy} = \mu_c$ (du/dy) might hold in the main body of the gas, up to a distance of the order of one local mean free path from a solid surface. By considering the balance of tangential momentum at the surface itself, Maxwell found that

$$u_{gas} - u_{wall} \cong \left(\frac{2\mu_c}{\rho\tilde{c}} \frac{du}{dy} \right)_{wall} ,$$

for completely diffuse reemission. According to kinetic theory, $\mu_c = (a/2)\,\rho\,\tilde{c}\,\lambda$, where $\tilde{c} = \sqrt{(8\mathcal{R}T/\pi)}$, and a is a numerical factor of order unity, so that $u_g - u_w \cong a\left(\lambda \frac{du}{dy} \right)_w$. It is interesting to compare this simple and widely-used suggestion with the results obtained from the present approximate solution of the Maxwell-Boltzmann equation.

When $M^2 << 1$, $p_{xy} \rightarrow \mu_c$ (du/dy) everywhere, according to the present solution (Sections II. B and III. B). At the upper plate [Eqs. (21a), (45c), and (45e)] ,

$$p_{xy}\,(+\,d/2) = \left(\rho_{II}\sqrt{\frac{\mathcal{R}T_{II}}{2\pi}}\; U \right) \left[\frac{\alpha_2}{\sqrt{T_I/T_{II}}} \left(\tfrac{1}{2} - \bar{u} \right) \right] = \left(\mu_c \frac{du}{dy} \right)_{y=+d/2} .$$

But $(\alpha_2/2) = \rho T/\rho_{II} T_{II}$ [Eq. (28c)] ; therefore,

$$(U/2) - u\,(+d/2) = \frac{2\mu_c}{\rho\,\tilde{c}} \sqrt{T_I/T}\;(du/dy)_{y=+d/2} = a\lambda\sqrt{T_I/T}\;(du/dy)_{y=+d/2} ,$$

where $(\rho \tilde{c})$ and μ_c, and λ <u>are evaluated at the gas temperature</u> (or the gas density), and not the surface temperature.

One sees that even for $M^2 << 1$ the Maxwell velocity slip relation is strictly correct only when the gas temperature and the surface temperature are nearly equal, i.e., when $(T_I - T_{II})/T_I << 1$ (Reference 10), <u>or</u> when $Re/M >> 1$. But

$$\sqrt{T_I/T} \cong 1 + \tfrac{1}{2}\left(\frac{\Delta T}{T_I - T_{II}}\right)\left(\frac{T_I - T_{II}}{T_I}\right) + \ldots ,$$

where $\Delta T = T_I - T(d/2)$. Now, $\Delta T/(T_I - T_{II}) \sim 0.10$ when $(Re/M)^* = 30$, so that the Maxwell velocity slip relation is in error by 5 per cent when $T_I/T_{II} = 2$ and $(Re/M)^* = 30$, or $(\lambda/d)^* = 1/20$. Thus the usual velocity slip relation is quite useful in the near-Navier-Stokes regime, as Maxwell suggested.

For arbitrary Mach number $p_{xy} \neq \mu_c\,(du/dy)$, and $(a_2/2) \neq \rho T/\rho_{II} T_{II}$. In that case the velocity slip can be expressed as follows:

$$(U/2) - u\,(+\,d/2) = a\left[\lambda \frac{\delta}{\sqrt{T_2/T}}\;(du/dy)\right]_{y=+d/2} ,$$

where

$$\delta = \frac{p_{xy}}{\mu_c\,(du/dy)} .$$

By utilizing Eq. (44), Section III. B, one can show that for $Re/M >> 1$,

$$\delta = 1 - \mathcal{s}_1\,(Re/M)^{-2} + \ldots. ;$$

also, $T_2/T = 1 + \mathcal{s}_2\,(Re/M)^{-1}$ where $\mathcal{s}_1$ and $\mathcal{s}_2$ are constants. Once again the Maxwell velocity slip relation is found to be correct to first order $\left[(Re/M)^{-1}\right]$ for large Re/M. The departure from the Navier-Stokes relation is of second order. We conclude that Maxwell's suggestion is a good first-approximation for arbitrary Mach number and plate temperature ratio in the near-Navier-Stokes regime. One could not expect it to hold for the transitional or highly rarefied flow regimes.

III. E. Limitations of the Six-Moment Approximation

Our original choice of six Maxwell moments (Section II. A) is expected to furnish a good first approximation for plane Couette flow when $M = 0(1)$. But at high plate Mach numbers ($M^2 >> 1$) p_{xx} and p_{yy} are of the same order as q_y in a rarefied gas [Eqs. 21 b, c, e, and f], and the six-moment approximation is inadequate.

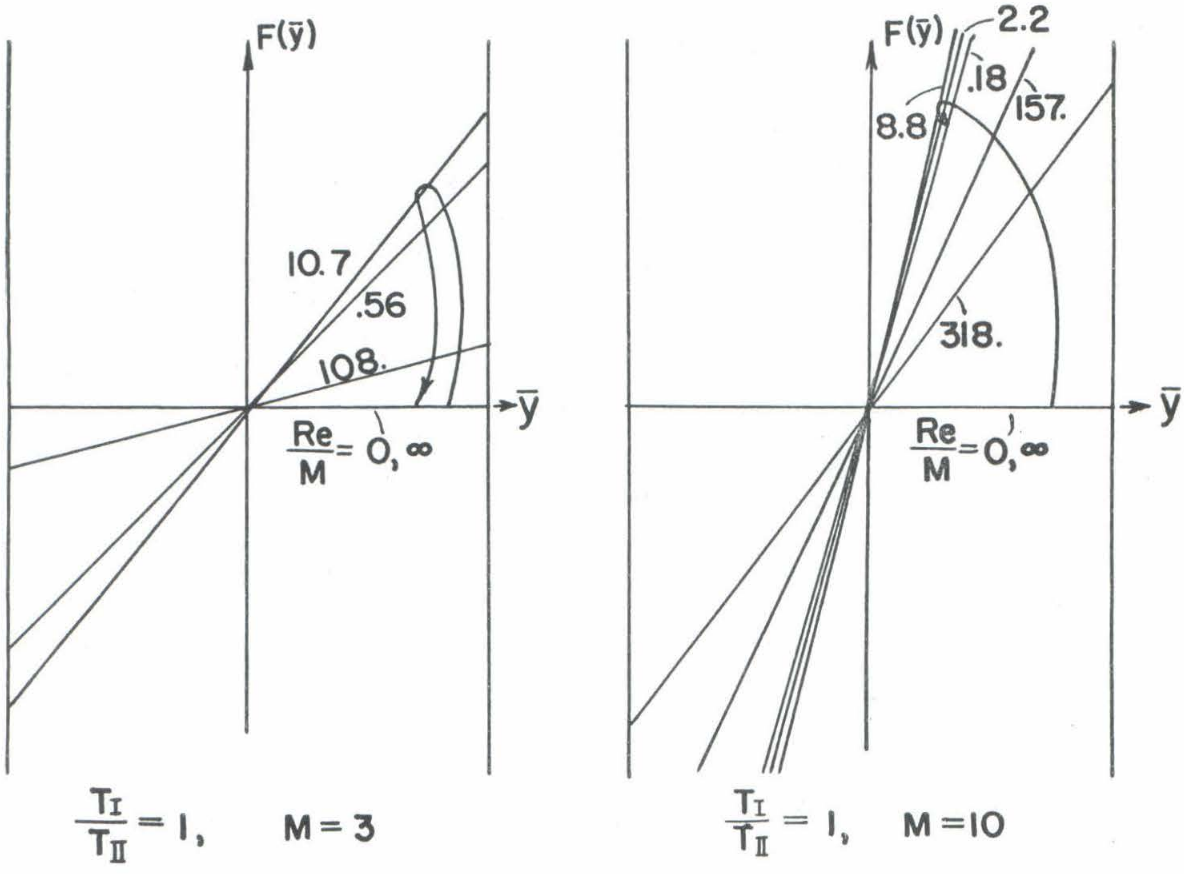
As an indication of the limitations of this approximation, consider the differential equation for (dF/dy) [Eq. (34)]:

$$dF/d\overline{y} = - \frac{8}{\sqrt{2\pi\gamma}} \frac{a_1}{(a_2^2 G^2 - F^2)} \left[\left(- \frac{4}{\gamma M^2} \frac{a_2}{a_1} + \frac{a_1}{a_2} \right)^{-1} \right] . \qquad (46)$$

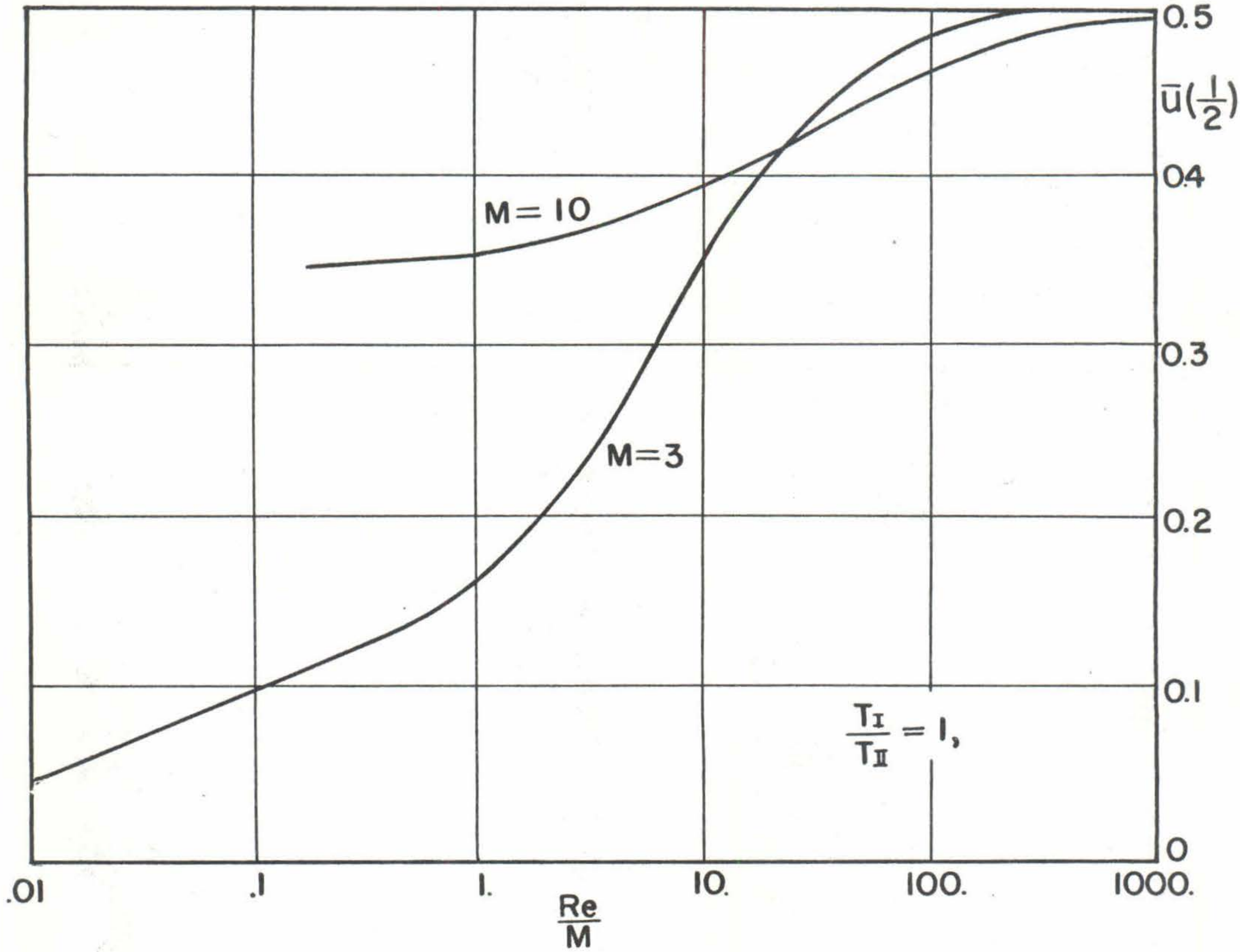
When $Re/M >> 1$, $a_2 \longrightarrow 2$ and $a_1 = 0\left[(Re/M)^{-1}\right]$, so the quantity in brackets is positive and $0\left[(Re/M)^{-1}\right]$. On the other hand when $Re/M \longrightarrow 0$, $a_2 = \overline{n}_1\overline{T}_1 + \overline{n}_2\overline{T}_2 \longrightarrow (1 + \sqrt{T_I/T_{II}})$, and $a_1 \longrightarrow -1$, so the bracket $\longrightarrow (1 + L)\left[\frac{4}{\gamma M^2}(1 + L)^2 - 1\right]^{-1}$, where $L = \sqrt{T_I/T_{II}}$. So long as $M^2 < (4/\gamma)(1 + L)^2$ this bracket is positive when $Re/M = 0$, and remains positive for all values of Re/M. However, when $M^2 > (4/\gamma)(1 + L)^2$ the bracket is negative in the limit $Re/M \longrightarrow 0$, and therefore must have changed sign for some intermediate value of Re/M. Such behavior is physically unrealistic, and some difficulties are to be expected with the six-moment approximation. (For $\gamma = 5/3$ and $L = 1$, the "critical" Mach number is 3.1.)

Without going into details we indicate the actual behavior of the curve of $F(\overline{y})$ vs. $\overline{y}$ and the behavior of $\overline{u}$ $(+ \frac{1}{2})$ for values of M^2 smaller and larger than $(4/\gamma)(1 + L)^2$. As specific examples, we selected $L = 1$

and M = 3 and 10. When $M^2 < (4/\gamma)(1 + L)^2$ the curve of $F(\bar{y})$ behaves in the manner described in Section II. C, and shown in the accompanying Sketch A. The value of $\bar{u}$ ($+\frac{1}{2}$) decreases smoothly as Re/M decreases, and the point of inflection shown in Sketch B occurs at a value of Re/M very close to the point at which the curve of $F(\bar{y})$ has its maximum inclination. However, for M = 10 the curve of $F(\bar{y})$ rotates counter clockwise to a certain maximum angle as Re/M decreases, but is unable



SKETCH A -- BEHAVIOR OF $F(\bar{y})$



SKETCH B -- BEHAVIOR OF $\bar{u}$ ($\frac{1}{2}$)

to negotiate the return journey to the horizontal position. In fact for values of Re/M less than a certain critical value no real solutions could be found (Sketch A and B).

The situation is somewhat analogous to the difficulty encountered with the Kármán-Pohlhausen method (14) when a quartic is employed to approximate the mean velocity profile across the laminar boundary layer. For positive streamwise pressure gradients it is well-known that flow separation occurs at $\Lambda = -12$, where Λ is the Pohlhausen parameter. But there is also a difficulty at $\Lambda = +12$, where none is expected on physical grounds. As shown by Tani (13) the best way to avoid (or postpone) such singularities is to take an additional moment. In our case the importance of the moments associated with p_{xx} and p_{yy} at high Mach numbers dictates a similar procedure.

A rough calculation replacing $Q_5 = m\,\xi_x \xi_y$ by $Q = m\,\xi_x^{\;2}$ (corresponding to p_{xx}) already shows that the difficulty illustrated by Eq. (46) dissappears. Of course this choice of moments is poor when $M = 0$ (1). Clearly the proper course is to employ an eight-moment approximation, in which the four moments in addition to the collisional invariants are as follows:

$$
\begin{aligned}
Q_5 &= m\,\xi_x \xi_y \\
Q_6 &= m\,\xi_y\,(\xi^2/2) \\
Q_7 &= m\,\xi_x^{\;2} \\
Q_8 &= m\,\xi_y^{\;2} \qquad .
\end{aligned}
$$

For this calculation we select a modified two stream Maxwellian of the following form:

$$
\begin{aligned}
\xi_y &< 0\,,\; f = f_1\left[1 + a_1(\bar{y})\;\xi_x\;\xi_y\right] \\
\xi_y &> 0\,,\; f = f_2\left[1 + a_2(\bar{y})\;\xi_x\;\xi_y\right],
\end{aligned}
$$

where f_1 and f_2 are given by Eqs. (10a) and (10b), and $a_1(\bar{y})$, $a_2(\bar{y})$ are two additional functions of y. The boundary conditions [Section II. A. 2] lead to the conditions given by Eq. (20), plus $a_1(+\frac{1}{2}) = 0$ and $a_2(-\frac{1}{2}) = 0$. An eight-moment approximation yields four algebraic and four first-order non-linear differential equations for the eight unknown functions $n_1(\bar{y})\,\ldots\ldots\ldots\,\bar{u}_2(\bar{y})$, $a_1(\bar{y})$, $a_2(\bar{y})$. Thus the problem is completely formulated, and is currently being investigated.

III. F. Conclusions and Future Work

By employing the simple two-stream Maxwellian in Maxwell's moment equations one obtains considerable insight into the nature of the transition from highly rarefied flows to the classical Navier-Stokes regime. The results obtained for plane, compressible Couette flow suggest certain conclusions about hypersonic flow. For a blunt-nosed body with surface temperature much less than the kinetic temperature the tangential velocity slip is expected to be very small near the nose, even in free-molecule flow. In spite of this fact the classical Navier-Stokes relations are not likely to provide a correct description of the flow field when $\lambda^*/R_0 > 1/20$ (approximately), where λ^* is the mean free path evaluated just behind the bow shock, and R_0 is nose radius. On the other hand the transition from the near-Navier-Stokes regime to nearly-free molecule flow occurs over a range of gas density of about one decade. Similar conclusions apply to those portions of slender bodies where the normal component of flight velocity is large compared with the thermal velocity corresponding to the surface temperature.

There are important differences between the present results and those obtained by the ad hoc procedure of utilizing the Navier-Stokes equations plus Maxwell's velocity slip relation (Section III. D). In spite of this fact, the values of skin-friction and heat transfer coefficients obtained by such an ad hoc procedure are not far wrong. As pointed out to the authors by Dr. H. Grad, plane Couette flow is probably still too simple a geometry to show any critical features of these gross macroscopic quantities. For this reason we are studying the problem of heat conduction between two concentric cylinders, where the ad hoc procedure is grossly inaccurate (1).

Another important example of non-linear flow is the steady, plane shock wave. This problem is also being investigated in order to learn about molecular effects in longitudinal flows without shear. Eventually one should have a much clearer understanding of the limitations and advantages of Maxwell's moment method.

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APPENDIX

PLANE COMPRESSIBLE COUETTE FLOW ACCORDING TO THE CLASSICAL NAVIER-STOKES EQUATIONS

The classical Navier-Stokes solution is given here for reference. In obtaining the solution, the medium is assumed to be a perfect gas, and the viscosity coefficient μ_c is directly proportional to the absolute temperature, just as for Maxwell molecules. Also, the Prandtl number is constant.

Clearly, the continuity equation $(d/dy)(\rho v) = 0$, together with the requirement that v vanishes at the plate surfaces leads immediately to

$$v \equiv 0 \quad . \tag{A-1}$$

Thus the conservation equations are as follows:

Momentum

$$(d/dy)\left(\mu_c \frac{du}{dy}\right) = 0 \tag{A-2}$$

$$(dp/dy) = 0 \tag{A-3}$$

Energy

$$(d/dy)\left[\frac{k_c}{c_p}\frac{d}{dy}(c_p T)\right] + \mu_c (du/dy)^2 = 0 \quad . \tag{A-4}$$

In addition, we have $p = \rho RT$. The corresponding boundary conditions are as follows:

$$y = -(d/2),\ u = -(U/2),\ T = T_{II};\ y = +(d/2),\ u = +(U/2),\ T = T_I. \tag{A-5}$$

Integration of Eqs. (A-2) and (A-4) yields the following momentum and energy integrals:

$$\mu_c\,(du/dy) = b_1$$

$$c_p\,T + (Pr/2)\,u^2 - (b_2/b_1)\,u = b_3 \quad ,$$

with the undetermined constants b_1 , b_2 , b_3 .

Let $(\mu_c)_{II}$ denote the viscosity coefficient evaluated at temperature T_{II}. Then by integrating Eq. (A-6) once again, we have

$$\int \mu_c/(\mu_c)_{II}\; du = \frac{b_1 y + b_4}{(\mu_c)_{II}} \quad .$$

But $\mu_c/(\mu_c)_{II} = T/T_{II}$, and by using Eq. (A-7) and integrating, we finally obtain

$$1/(c_p\,T_{II})\left[b_3\,u + (b_2/2b_1)\,u^2 - (Pr/b)\,u^3\right] = \frac{b_1}{(\mu_c)_{II}}\,y + \frac{b_4}{(\mu_c)_{II}} \quad ,$$

where b_4 is another integration constant. The four b's appearing in Eqs. (A-7) and (A-8) are determined by the four boundary conditions [Eqs. (A-5)] . The final results expressed in terms of non-dimensional quantities are

$$C_D = (p_{xy})/(\tfrac{1}{2}\rho_{II}U^2) = (1/Re)\left[1 + (T_I/T_{II}) + \frac{\gamma-1}{6}\,Pr\,M^2\right] \qquad \text{(A-9)}$$

$$C_H = \tfrac{1}{2}\left[\frac{1}{Pr\,Re} - \frac{\gamma-1}{2}\,\frac{T_{II}}{T_{II}-T_I}\,\frac{M^2}{Re}\right]\left[1 + \frac{T_I}{T_{II}} + \frac{\gamma-1}{6}\,Pr\,M^2\right] . \qquad \text{(A-10)}$$

The velocity and temperature profiles are given by the relations

$$\left[1 + (T_I/T_{II}) + \frac{\gamma-1}{4}\,Pr\,M^2\right](u/U) - \left(1 - \frac{T_I}{T_{II}}\right)(u/U)^2 - \frac{\gamma-1}{3}\,Pr\,M^2\,(u/U)^3$$

$$= \left[1 + (T_I/T_{II}) + \frac{\gamma-1}{6}\,Pr\,M^2\right]\bar{y} - 1/4\left(1 - \frac{T_I}{T_{II}}\right) \qquad \text{(A-11)}$$

$$\frac{T}{T_{II}} + \frac{\gamma-1}{2}\,Pr\,M^2\,(u/U)^2 + \left(1 - \frac{T_I}{T_{II}}\right)\frac{u}{U} = \tfrac{1}{2}\left(1 + \frac{T_I}{T_{II}}\right) + \frac{\gamma-1}{8}\,Pr\,M^2 \qquad \text{(A-12)}$$

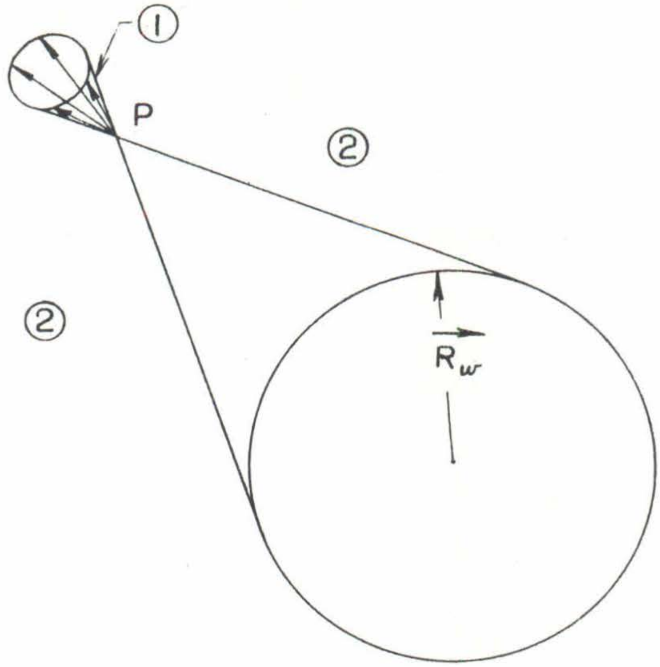

FIG. 1

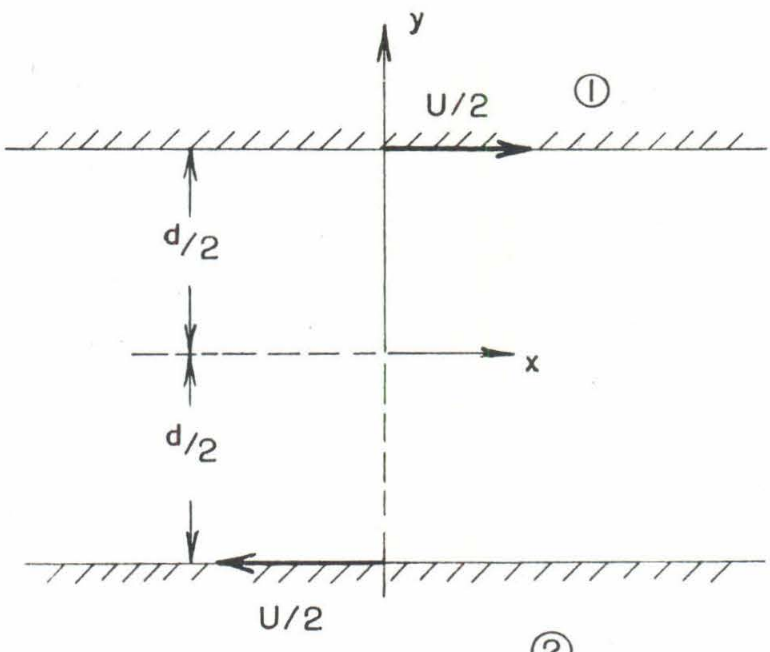

FIG. 2

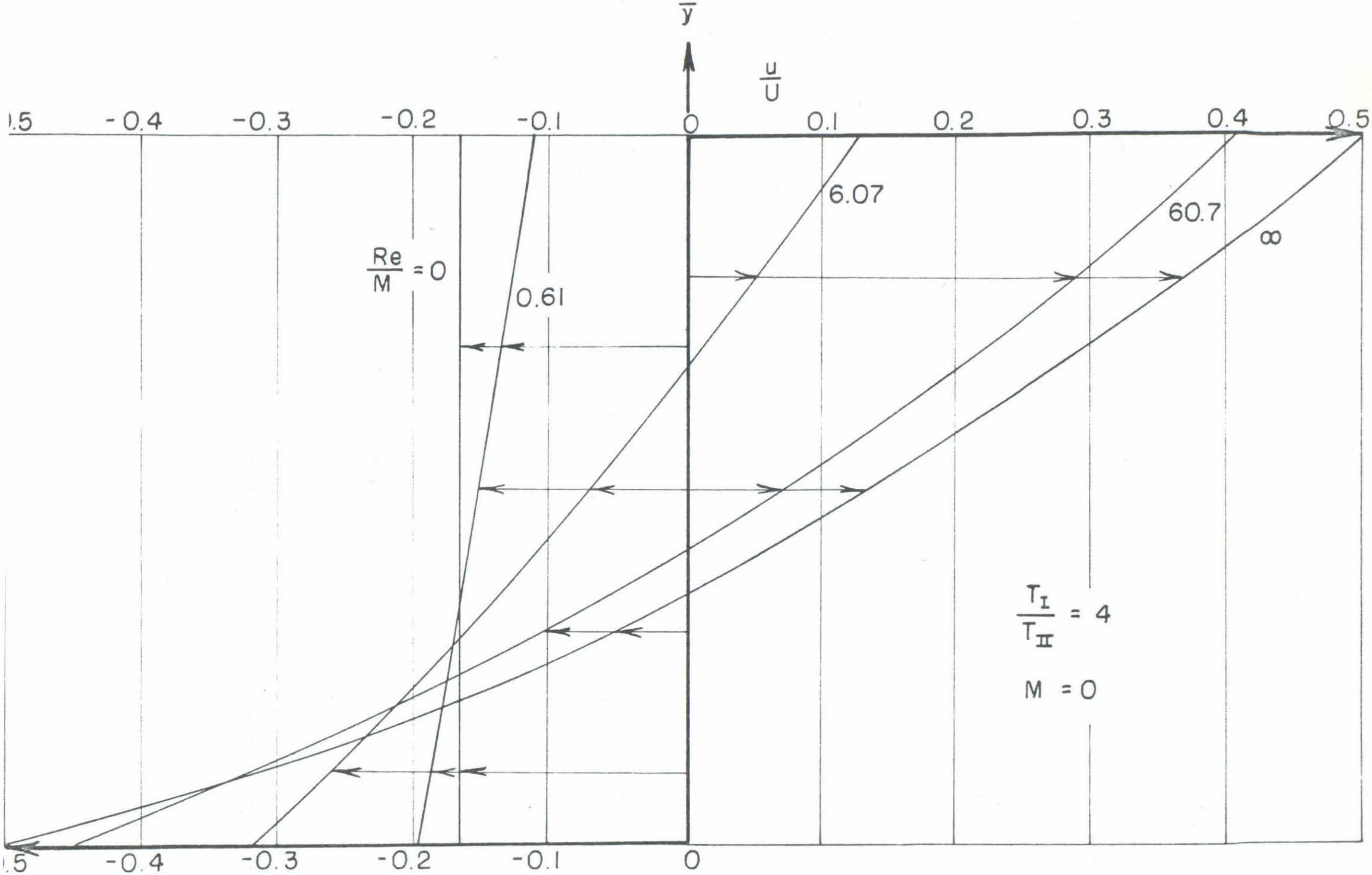

FIG. 3 - VELOCITY PROFILES FOR PLANE COUETTE FLOW

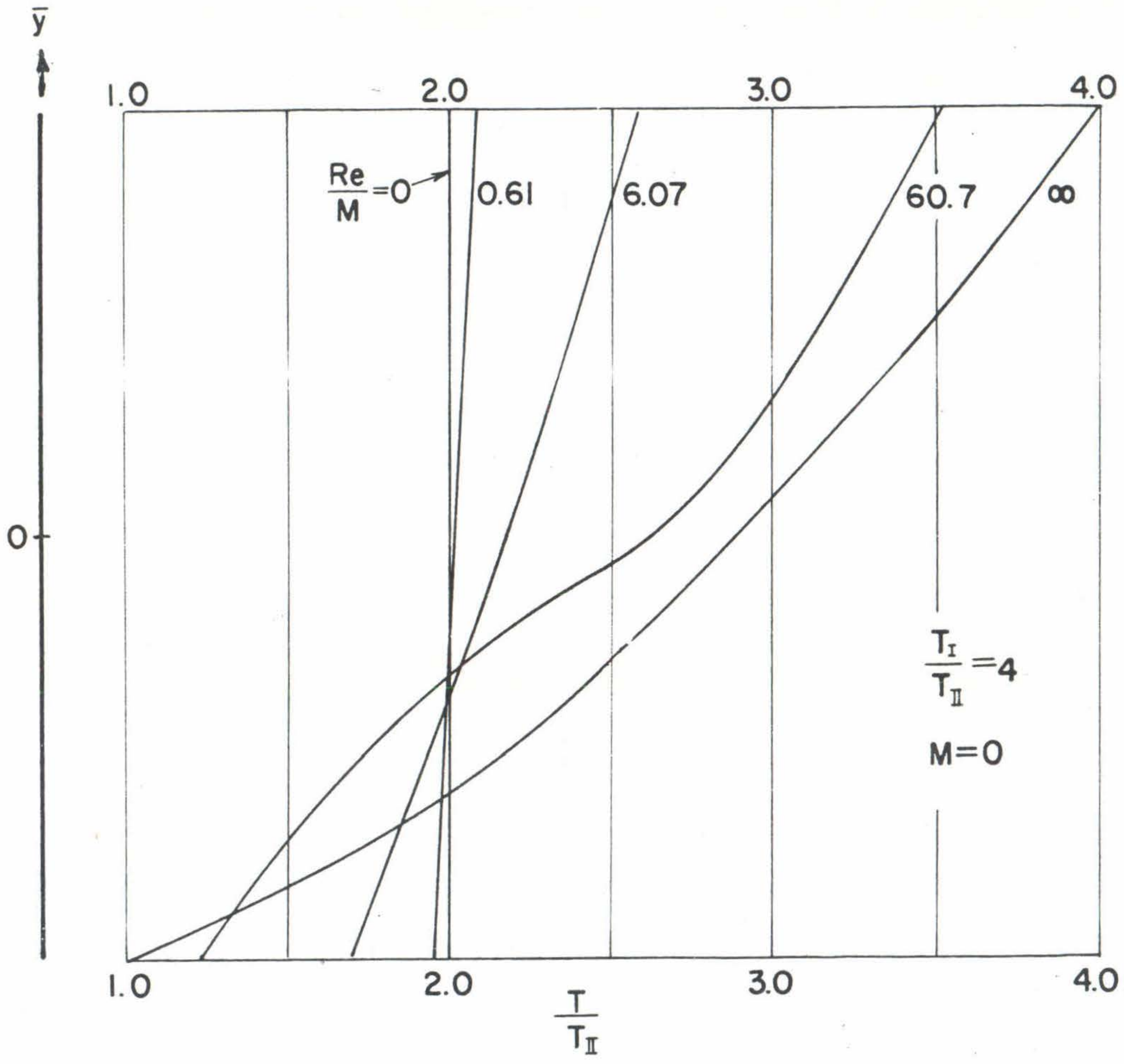

FIG. 4 TEMPERATURE PROFILES FOR PLANE COUETTE FLOW

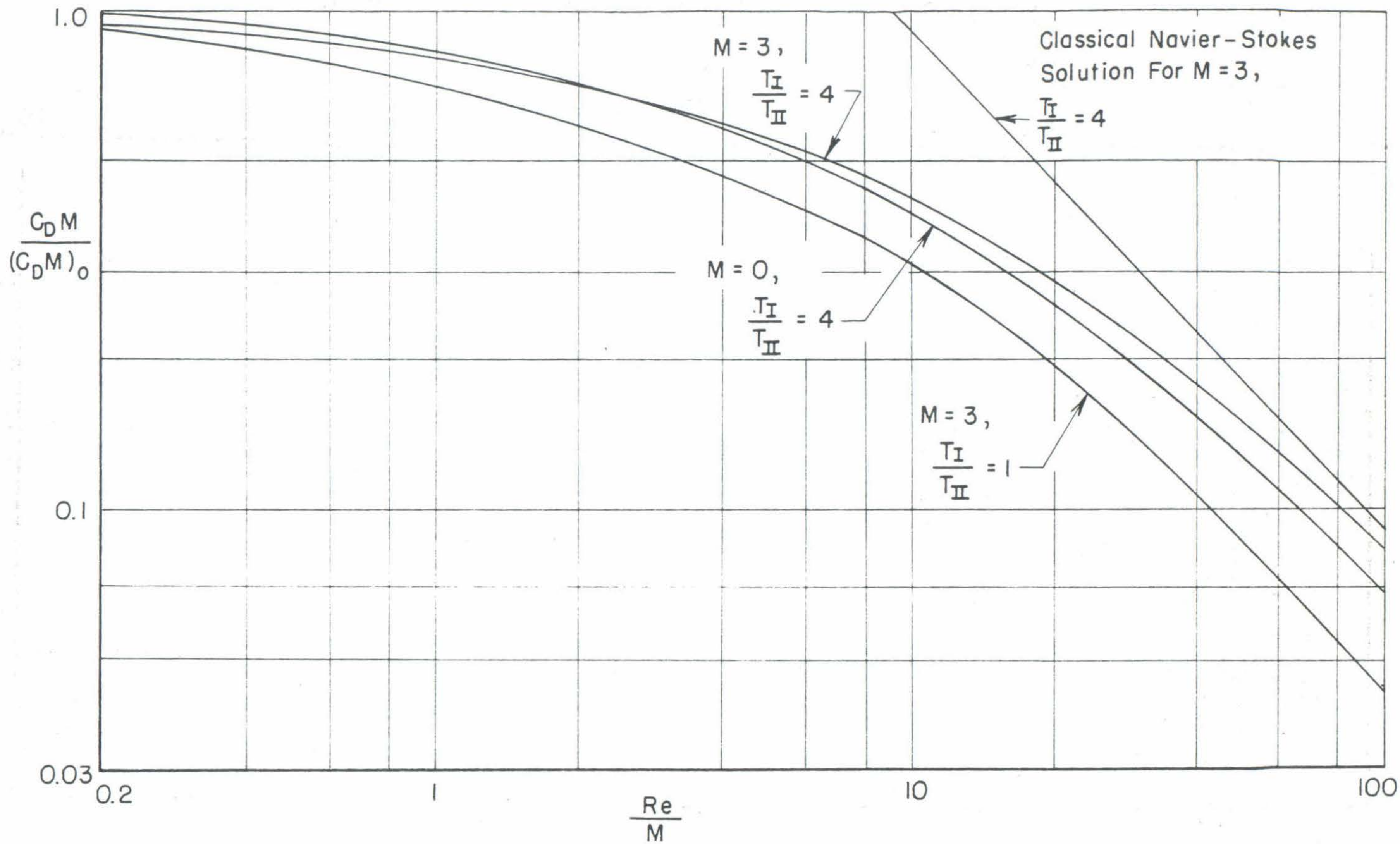

FIG. 5 - SKIN FRICTION IN PLANE COUETTE FLOW

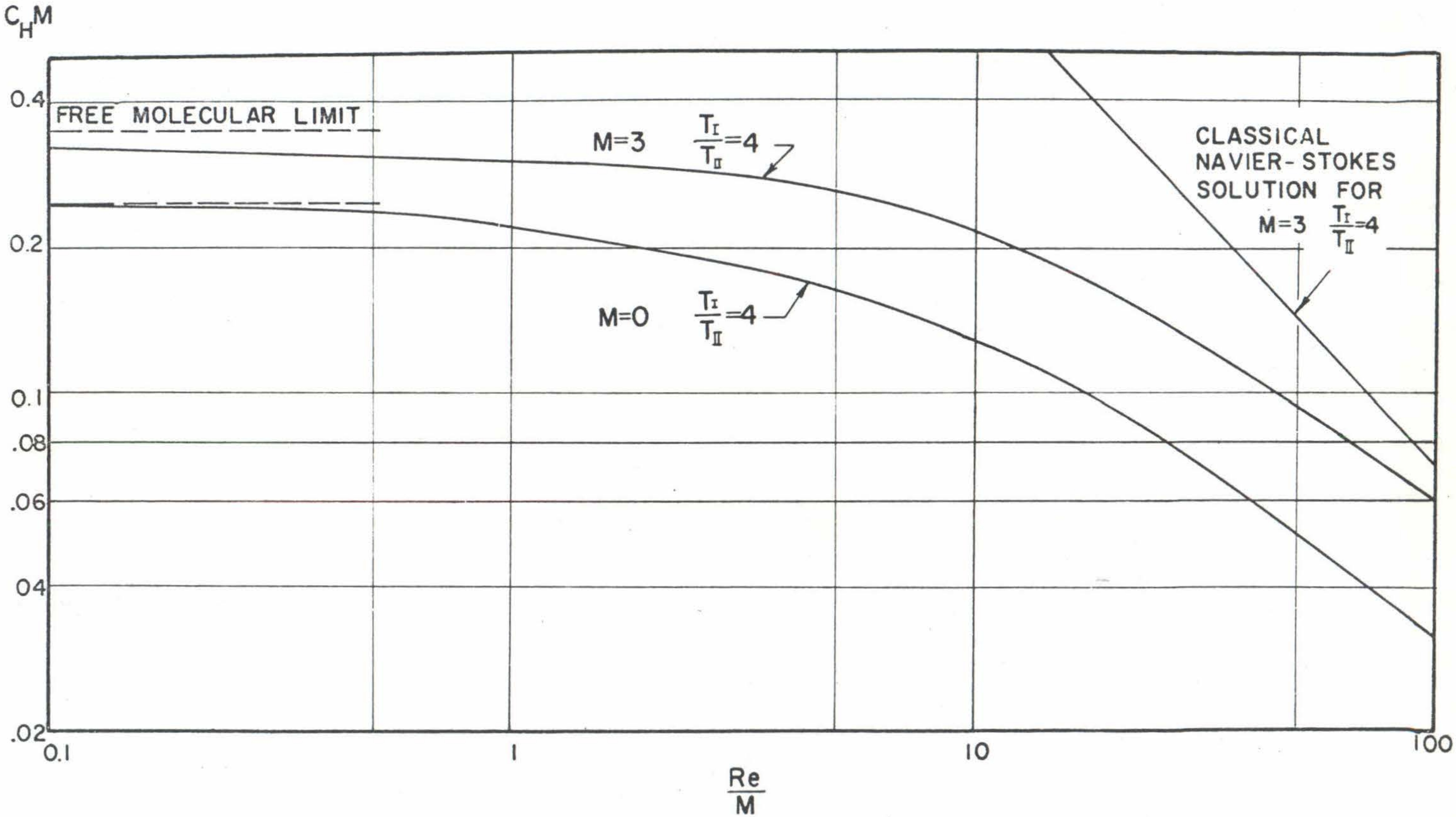

FIG. 6 HEAT TRANSFER IN PLANE COUETTE FLOW

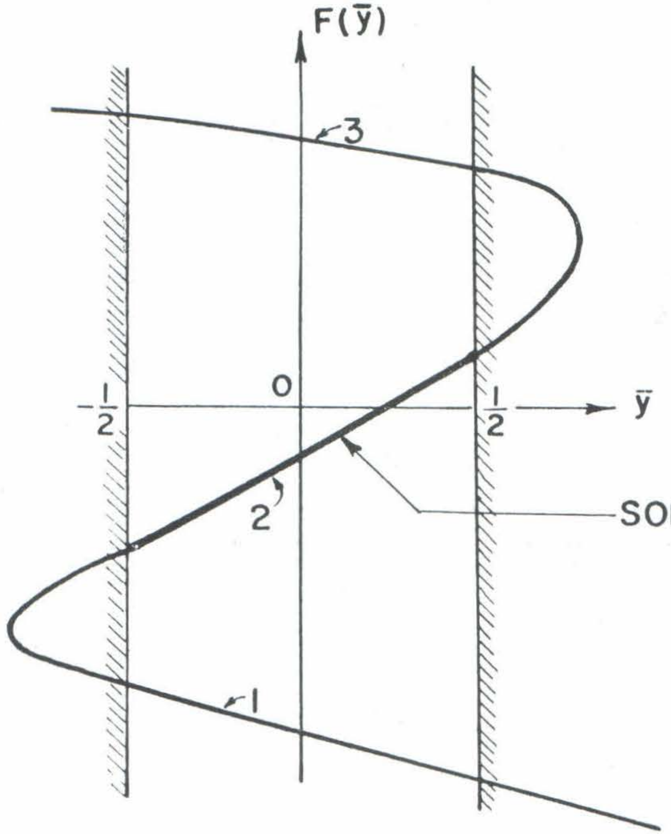

FIG. 7 TYPICAL VARIATION OF F($\bar{y}$)

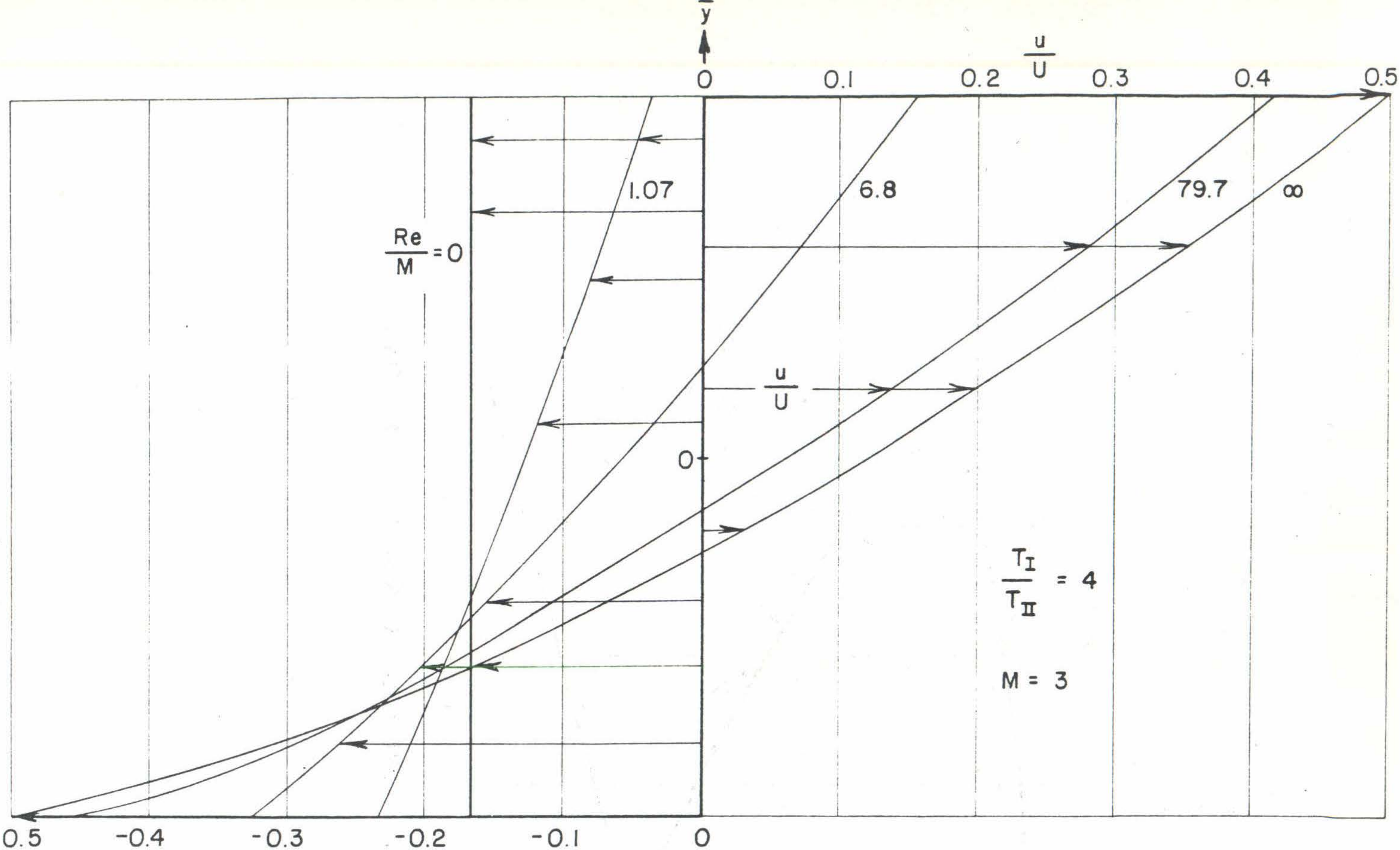

FIG. 8 – VELOCITY PROFILES FOR PLANE COUETTE FLOW

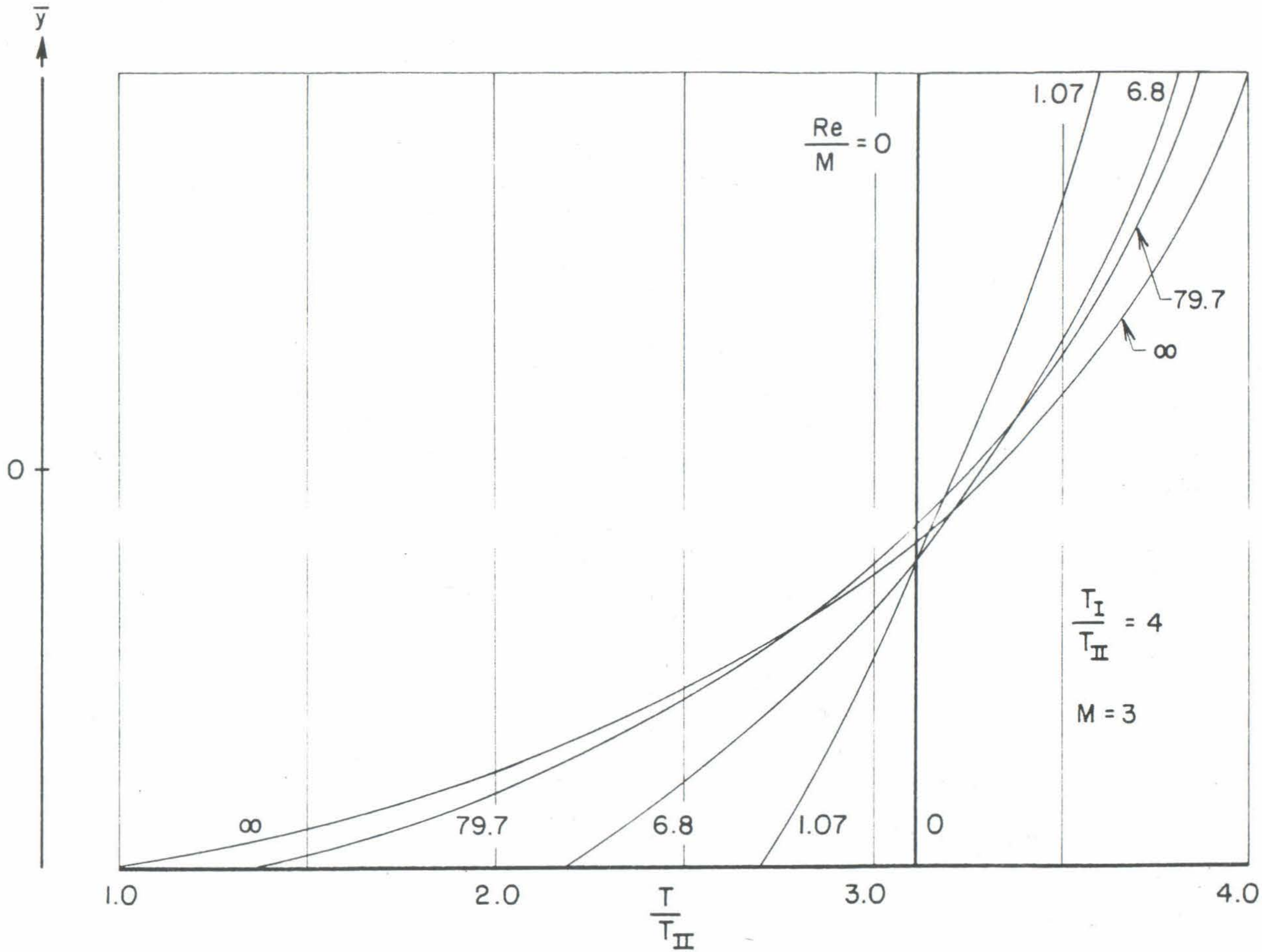

FIG. 9 - TEMPERATURE PROFILES FOR PLANE COUETTE FLOW

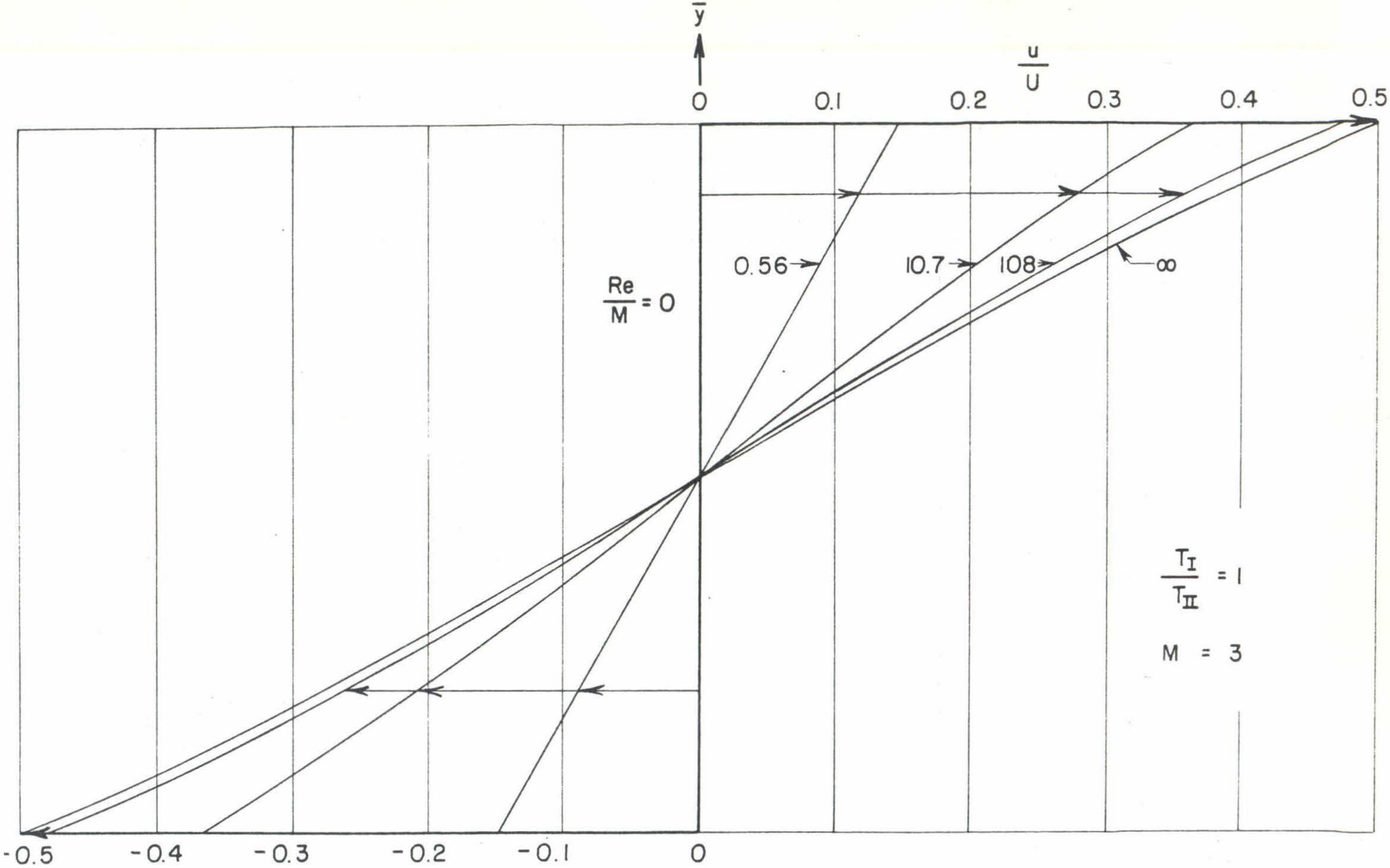

FIG. 10 - VELOCITY PROFILES FOR PLANE COUETTE FLOW

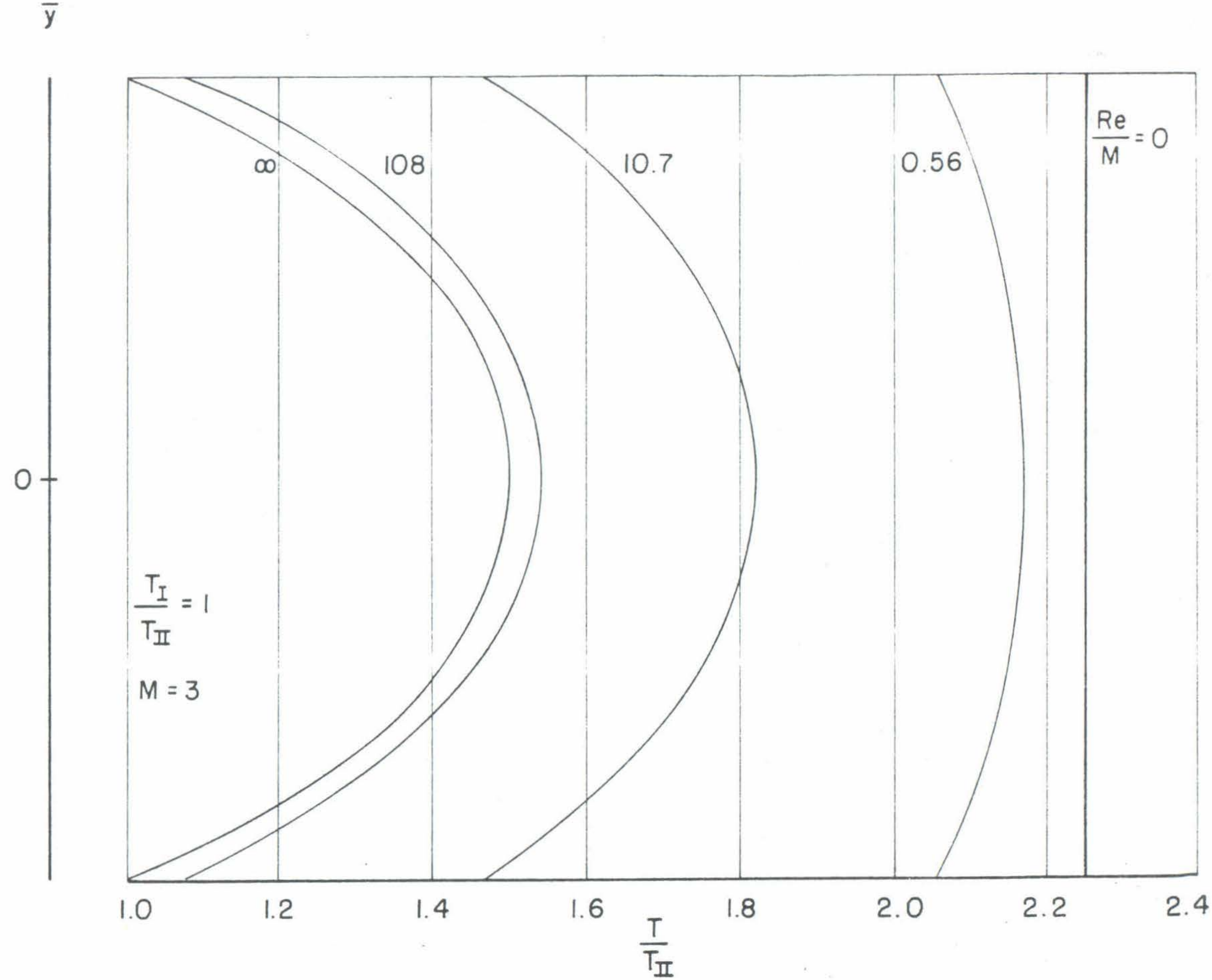

FIG. 11 - TEMPERATURE PROFILES FOR PLANE COUETTE FLOW

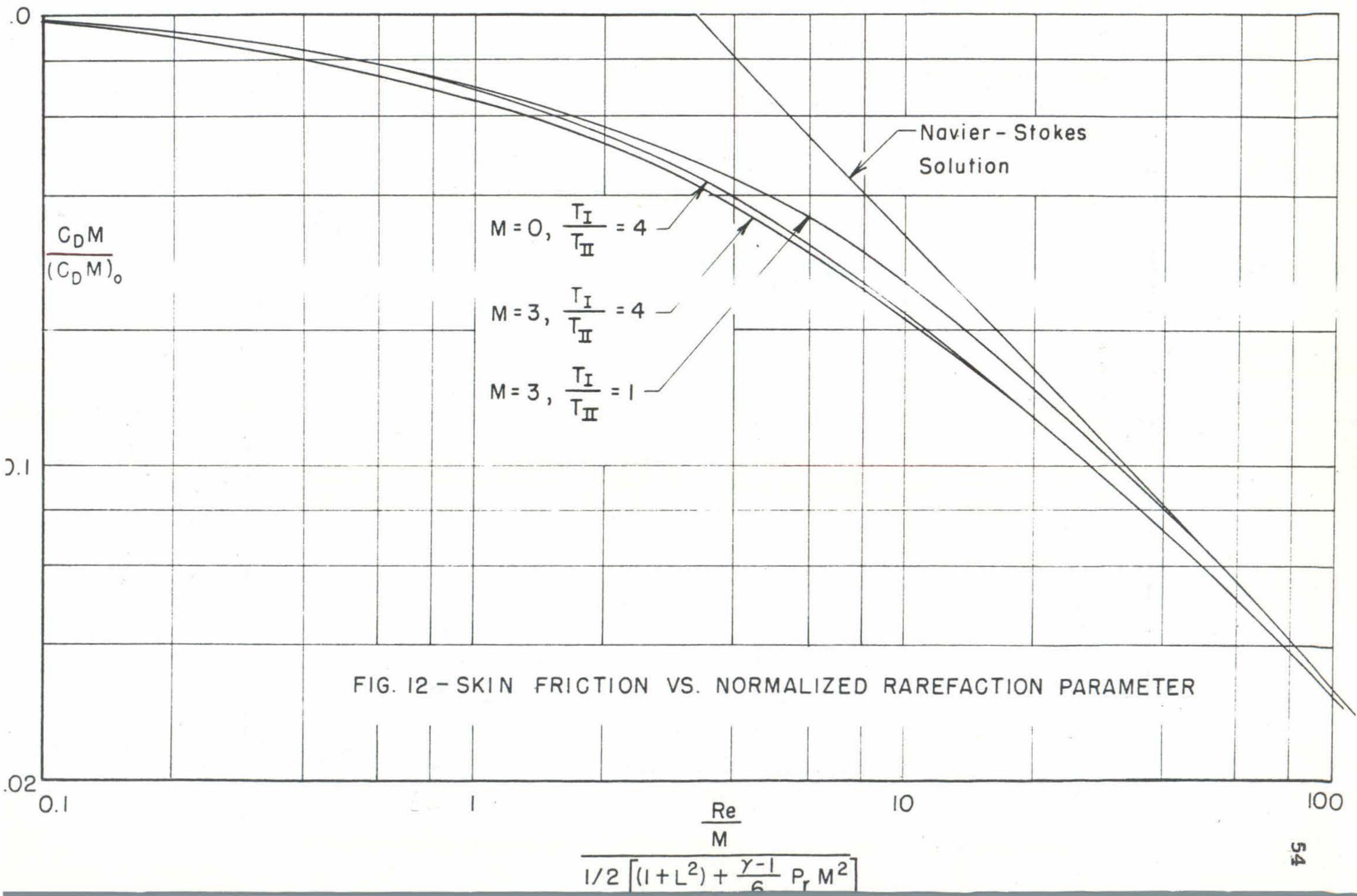

FIG. 12 - SKIN FRICTION VS. NORMALIZED RAREFACTION PARAMETER

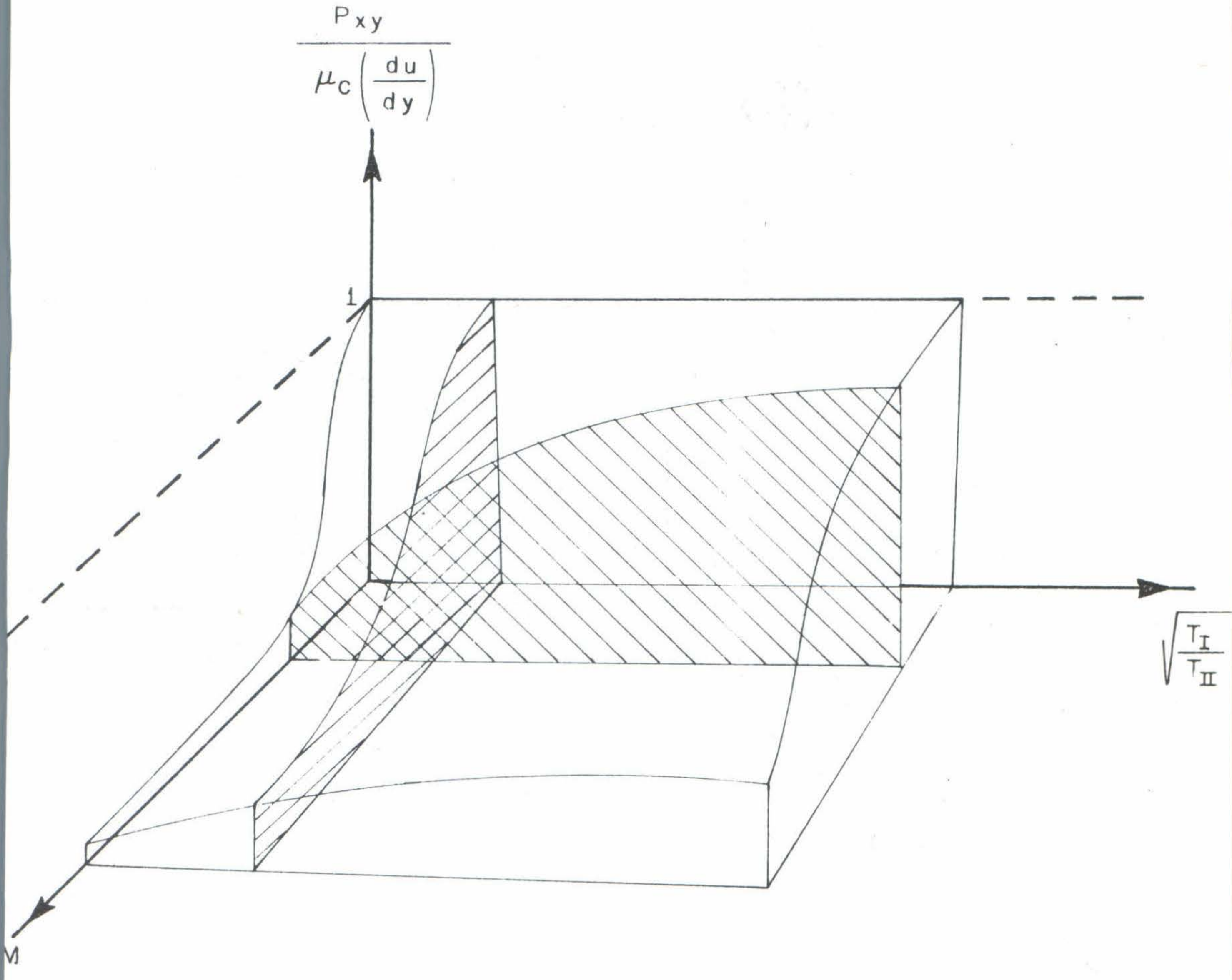

FIG. 13—DEPARTURE FROM NAVIER-STOKES RELATION IN PLANE COUETTE FLOW. $\frac{Re}{M} = 0$

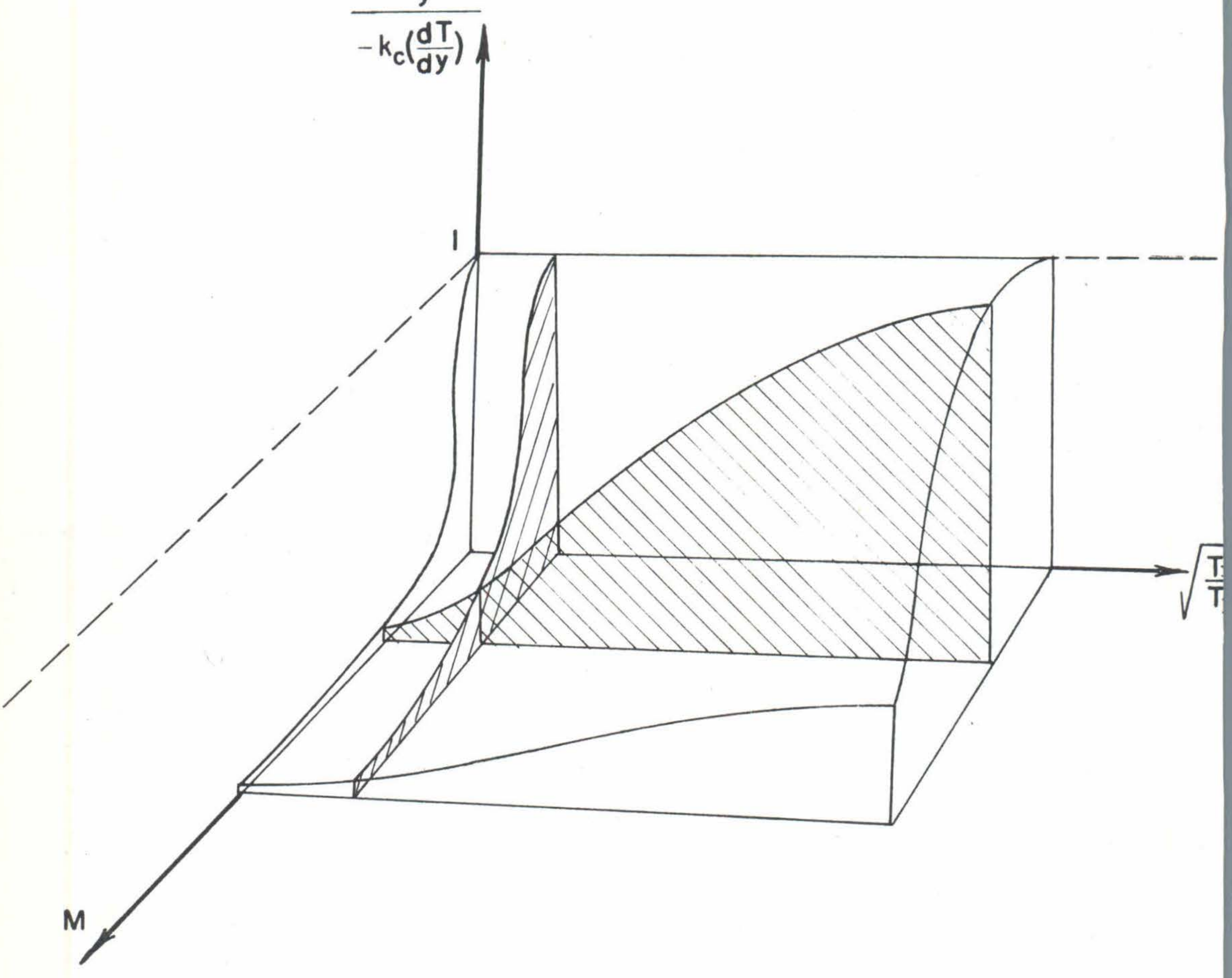

FIG. 14 — DEPARTURE FROM FOURIER'S RELATION IN PLANE COUETTE FLOW. $\frac{Re}{M}=0$

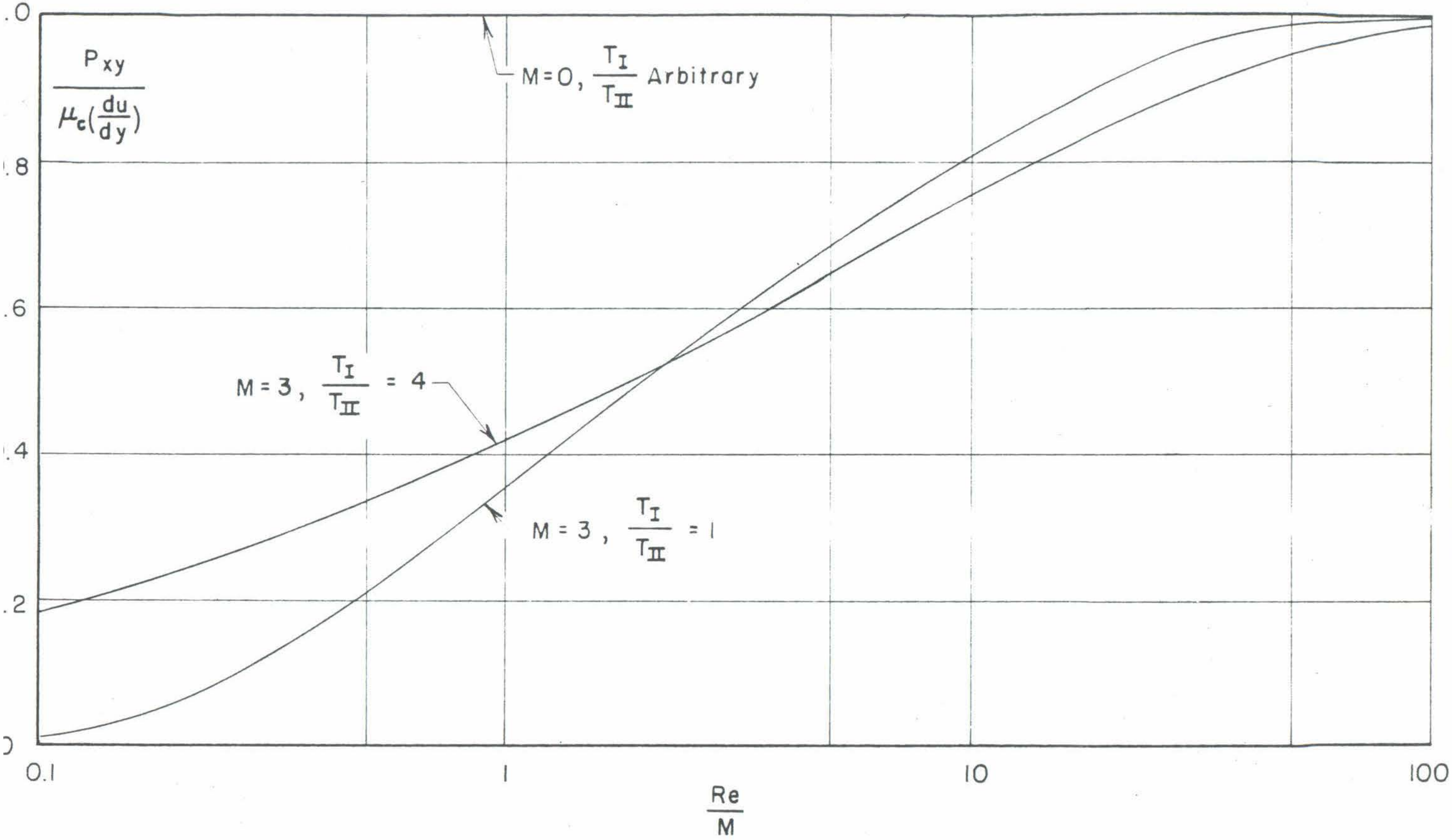

FIG. 15 – DEPARTURE FROM NAVIER – STOKES RELATION AS A FUNCTION OF $\frac{Re}{M}$

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